A COMPLEX POMPEIU PROBLEM

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1. Introduction. This paper has its origin in two results of the Rumanian mathematician Dimitrie Pompeiu. Let B be a circle of fixed radius, and suppose that $f(x, y)$ is a continuous function of the real variables x and y. Suppose, now, we know that

$$\iint_B f(x, y)\, dx\, dy = 0 \tag{1}$$

for all positions of the circle B. Does it follow that $f(x, y) \equiv 0$? Pompeiu, [13] or [15; 335], thought it did. Some years later, it became apparent that this conclusion was false; indeed, for appropriate constants a and b, the function $f(x, y) = \sin (ax + by)$ will satisfy equation (1) for all positions of the circle B. On the other hand, Pompeiu also proved, [12] or [15; 345], that if B designates a *square* with a fixed side, and if $f(x, y)$ is continuous and has a unique limit as $x^2 + y^2 \to \infty$, then (1) holds for all positions of B if and only if $f(x, y) \equiv 0$. Christo Christov, a Bulgarian mathematician, succeeded in establishing Pompeiu's result without the restricting condition as to the behavior of f at ∞ [3]. In a later paper [4], Christov established a similar result if B is a parallelogram, a triangle, or a trapezoid.

Additional work in this area includes that of Pompeiu [14] and Nicolescu [10], [11].

What is the situation when $f(x, y)$ is replaced by $f(z)$, a regular function of a complex variable $z = x + iy$, and B is a fairly general region of the complex plane? If Ω is the set of all Euclidean transformations T of the plane, when does

$$\iint_B f(T(z))\, dx\, dy = 0, \qquad T \in \Omega_1 \tag{2}$$

for some set $\Omega_1 \subset \Omega$ imply $f = 0$? We call this the *complex Pompeiu problem* and present a number of theorems towards its solution.

We shall work more generally with integrals of the form $\iint_B f(T(z))\overline{g(z)}\, dx\, dy$. If B is a domain, the space of functions h regular on B such that

$$\iint_B |h(z)|^2\, dx\, dy < \infty$$

is a Hilbert space $L^2(B)$ with inner product defined by

$$(f, g) = \iint_B f(z)\overline{g(z)}\, dx\, dy.$$

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