EXTENSION OF FENCHEL'S DUALITY THEOREM FOR CONVEX FUNCTIONS

BY R. T. ROCKAFELLAR

1. Introduction. Let E be a locally convex Hausdorff topological vector space over the real numbers R with dual E^*. Let f be a proper convex function on E, i.e. an everywhere-defined function with values in $(-\infty, +\infty]$, not identically $+\infty$, such that

$$(1.1) \qquad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ if } x \in E, \quad y \in E, \quad 0 < \lambda < 1.$$

Let g be a proper concave function on E (i.e. $-g$ is proper convex). This paper is concerned with characterizing the solutions and the extremum in the following problem:

$$\text{(I)} \qquad \text{minimize } f(x) - g(x) \text{ on } E.$$

Many constrained, as well as unconstrained, extremum problems can be represented in the model form (I), because the functions are allowed to be infinite-valued. For example, if D is a convex set in E and $g(x) = 0$ for $x \in D$, $g(x) = -\infty$ for $x \notin D$, then (I) is essentially the same as minimizing f on D.

Closely associated with (I) is a "dual" problem of similar type,

$$\text{(II)} \qquad \text{maximize } g^*(x^*) - f^*(x^*) \text{ on } E^*,$$

where the concave function g^* and the convex function f^* are the *conjugates* [2, 4, 7] of f and g defined by

$$(1.2) \qquad f^*(x^*) = \sup \{(x, x^*) - f(x)\},$$

$$(1.3) \qquad g^*(x^*) = \inf \{(x, x^*) - g(x)\}$$

for each $x^* \in E^*$. It is immediate from (1.2) and (1.3) that

$$(1.4) \qquad f(x) - g(x) \geq g^*(x^*) - f^*(x^*) \quad \text{for all} \quad x \in E \quad \text{and} \quad x^* \in E^*.$$

Problem (II) was first introduced (in the finite-dimensional case, and in a slightly different formulation) by Fenchel [5], who showed that (1.4) could often be strengthened to

$$\text{(A)} \qquad \inf \{f(x) - g(x)\} = \max \{g^*(x^*) - f^*(x^*)\}.$$

Fenchel's duality theorem [5; 108] asserts, namely, that (A) is true when $E = R^n$, if the relative interiors of the convex sets

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