SPECIAL CLASSES OF SUBORDINATE FUNCTIONS

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1. Introduction. The writing of this paper has been motivated by two recent papers [8], [5] of H. S. Wilf and M. S. Robertson. By combining the techniques used in these two papers we will obtain results, some of which are well known while others are new. For ease of reference we state here the principal results in [8], [5] that will be used in this paper.

Let γ denote the interior of the unit circle $|z| = 1$, and (S) the class of functions $f(z) = a_1z + a_2z^2 + \cdots$ which are regular and univalent in γ and map γ onto a simply-connected domain D. If the domain D is convex, then $f(z)$ is a member of the class (K). Let $g(z) = b_1z + b_2z^2 + \cdots$ be regular in γ, $f(z) \in (K)$. The notation $g(z) \prec f(z)$ ("$g(z)$ is subordinate to $f(z)$") will mean that every value taken by $g(z)$, $z \in \gamma$, is also taken by $f(z)$. We adopt from [8] the following definition.

Definition (A). An infinite sequence $\{c_n\}_1^\infty$ of complex numbers will be called a subordinating factor sequence if whenever $f(z) = a_1z + a_2z^2 + \cdots \in (K)$ we have $\sum_1^\infty a_nc_nz^n \prec f(z)$. (We will sometimes abbreviate $\{c_n\}_1^\infty = c_n = c(n)$.) The class of such sequences is denoted by (F). Wilf's principal result in [8] is contained in the following theorem.

Theorem (A). *The following three properties of a sequence of complex numbers are equivalent*:

(I) $\quad \{c_n\}_1^\infty \in (F)$

(II) $$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} c_nz^n\right\} > 0$$

(III) $$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\, d\psi(\theta) \qquad (n = 0, 1, 2, \cdots; c_0 = 1; \psi(\theta)\uparrow).$$

Among the various applications in [8] of Theorem (A) new proofs of the following two well-known results are given:

Corollary (A.1). *If* $\{b_n\}_1^\infty \in (F)$ *and* $\{c_n\}_1^\infty \in (F)$, *then* $\{b_nc_n\}_1^\infty \in (F)$.

Corollary (A.2). *The image of the unit circle by a function* $f(z) = z + a_2z^2 + \cdots \in (K)$ *contains the circle* $|w| < \frac{1}{2}$ (*the* $\frac{1}{2}$-*theorem*).

Robertson's principal result in [5] is contained in the following theorem.

Theorem (B). *Let* $f(z) = z + a_2z^2 + \cdots$ *be regular and univalent in* $|z| < 1$.

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