THE BAIRE ORDER PROBLEM FOR COMPACT SPACES

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Let C_β denote the algebra of all continuous real-valued functions on a compact (Hausdorff) space (X, β), with the algebra operations defined pointwise. The main results of the present paper show that the algebra of Baire functions generated by C_β has a particularly simple structure if and only if the topological space (X, β) is dispersed. (A dispersed space is one which has no non-void perfect subsets.)

More precisely, we show that there are only two classes of spaces (X, β). If (X, β) is dispersed, then every Baire function is the pointwise limit of a sequence of continuous functions (i.e., has Baire order ≤ 1). On the other hand, if, (X, β) is not dispersed, then, in the following sense, there exist Baire functions of every order: There is a uniformly closed subalgebra S of C_β such that for every countable ordinal η the η-th iteration of the sequential closure of S is not sequentially closed. The first case is studied in §2 by exploiting a method due to Rudin [8] for decomposing compact dispersed spaces. The second case is studied in §3 by constructing, in an arbitrary non-dispersed compact space, a "generalized Cantor set" and then utilizing the classical theorem for non-dispersed complete metric spaces.

Terminology and notation. The algebra of all real-valued functions on a set X is denoted by R^X; the subalgebra of all bounded functions is $(R^X)^*$. If τ is one of several topologies on X, the subalgebra of all τ-continuous functions is denoted by C_τ ; C_τ^* is the subalgebra of all bounded such functions. The sequence $f_1, f_2, \cdots$ is denoted by (f_n). In general, when no indexing set is specified, it is understood to be the set N of positive integers. Compact topological spaces are assumed to be Hausdorff.

1. Fundamental relations between the topologies. The *algebra of Baire functions* on the compact space (X, β) is defined as the smallest class of bounded functions containing $C_\beta(X)$ and closed under the operation of taking pointwise limits of sequences. Basic material on Baire functions can be found in Lorch [4]. We summarize briefly some results from [4, §5] that will be needed here. The weak topology on X generated by the Baire functions is called the *ι-topology*. Thus the Baire functions form a (possibly proper) subalgebra of $C_\iota^*(X)$. The zero sets of C_β form a basis for the *ι-open* sets. The ι-topology is

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