HOMEOMORPHISMS AND INVARIANT MEASURES FOR $\beta N - N$

By Ralph A. Raimi

Introduction. Let N be a discrete countable set (e.g. the positive integers); as a completely regular topological space N has a *Stone-Čech compactification* βN [4] and [8]. We shall be interested in the compact space $\hat{N} = \beta N - N$. If $s : N \to N$ is a $1:1$ mapping under which N has no periodic points, s will be called a *motion*, and it induces a homeomorphism $s : \hat{N} \to \hat{N}$. In §2 we describe the support sets of the s-invariant Borel probability measures on $\hat{N}$. Briefly, the results are that for a given motion s, the union of the supports of all s-invariant Borel measures is nowhere dense, but that the union of these unions, as s ranges over all motions, *is* dense in $\hat{N}$. If the Continuum Hypothesis is assumed, the complement of the last-named union is also dense. In §3 some of the measure theory and a theorem of W. Rudin are used to show that on no closed s-invariant subset of $\hat{N}$ is the sequence of homeomorphisms $\{s^n\}$ uniformly equicontinuous, whatever motion s be chosen. Motions are, in this sense, very different from isometries of a compact metric space. In §4 the result of §3 is extended to more general classes of homeomorphisms of $\hat{N}$. If the Continuum Hypothesis is assumed, *all* homeomorphisms of $\hat{N}$, except for those which are periodic, give rise to non-equicontinuous sequences of iterates. The proofs demand an investigation of the existence of enough points in $\hat{N}$ which are aperiodic under s.

1. Background. 1.1. If $A \subset N$ is any subset, $\hat{A}$ will denote the open-closed subset of N obtained as $\hat{N} \cap \bar{A}$, where $\bar{A}$ is the closure of A in βN. $\hat{A}$ is empty if and only if A is finite. Conversely, if $\hat{A}$ is a given open-closed subset of $\hat{N}$, there exists a subset $A \subset N$ such that $\hat{A} = \bar{A} \cap \hat{N}$. Such an A will be called *antecedent* to $\hat{A}$. If A and B are both antecedent to $\hat{A}$, then A and B agree except for a finite number of points.

Similarly, if f is a bounded real-valued function on N, and $\bar{f}$ its extension to βN, we denote by $\hat{f}$ the restriction of $\bar{f}$ to $\hat{N}$. Conversely, if $\hat{f}$ is given as a continuous real-valued function on $\hat{N}$, there exists a bounded function f on N such that $\hat{f}$ is obtainable from f in this way. We call f *antecedent* to $\hat{f}$, and observe that if f and g are both antecedent to $\hat{f}$, then $f - g$ vanishes at infinity in N.

1.2. Let $s : N \to N$ be a motion; as in [6], we denote by $\mathfrak{M}$ the class of all motions. Since s can be extended (uniquely) to a homeomorphism of βN into βN, under which N is an invariant set, s induces a homeomorphism of $\hat{N}$ into $\hat{N}$, which we shall continue to call s, and continue to call a *motion*. $s : \hat{N} \to \hat{N}$ is onto if and only if $N - sN$ (set-theoretic difference) is finite.

Received October 1, 1964. Research supported by National Science Foundation Grant G 23799.