# INVERSE SERIES RELATIONS AND OTHER EXPANSIONS INVOLVING HUMBERT POLYNOMIALS

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**1. Introduction.** The object of this paper is to present some novel expansions, in particular a pair of inverse series relations, which occur in the study of a general class of polynomials defined as follows. By a generalized Humbert polynomial $P_n(m, x, y, p, C)$ we mean the coefficient determined by the series expansion of $H = H(t; m, x, y, p, C) = (C - mxt + yt^m)^p$,

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C), \tag{1.1}$$

where $m \geq 1$ is an integer and the other parameters are unrestricted in general. That $P_n$ is a polynomial in $x$ will be made evident later by means of explicit formulas.

This definition includes many well-known and not so well-known special cases. We tabulate the main cases, each name being followed by a date and citation:

$$P_n(2, q, -1, -\tfrac{1}{2}, p^2) = f_n(p, q), \qquad \text{Louville (1722) [24]}$$

$$P_n(2, x, 1, -\tfrac{1}{2}, 1) = X_n = P_n(x), \qquad \text{Legendre (1784) [22]}$$

$$P_n(2, x, 1, -1, 1) = U_n(x), \qquad \text{Tchebycheff (1859) [32]}$$

$$P_n(2, x, 1, -\nu, 1) = C_n^\nu(x), \qquad \text{Gegenbauer (1874) [12]}$$

$$P_n(3, x, 1, -\tfrac{1}{2}, 1) = \text{“}P_n(x)\text{”}, \qquad \text{Pincherle (1890) [16]}$$

$$P_n(m, x, 1, -\nu, 1) = \textstyle\prod_{n,m}^{\nu} (x), \qquad \text{Humbert (1921) [16]}$$

$$P_n(m, x, 1, -1/m, 1) = P_n(m, x), \qquad \text{Kinney (1963) [20]}$$

The name Louville requires special comment since Truesdell [34; 86] speaks of "Gegenbauer's generalization of the Chevalier d'*Alouville* de Louville's classical generating function for the Legendre polynomials" and refers the reader to [24; 132]. In a personal communication (1964) Professor Truesdell has written that he first read about the work of de Louville in Nielsen's treatise [26] on French geometers of the 18-th Century. The basic facts surrounding the life and work of Louville are not too hard to come by, but some inconsistency exists about dates, place of birth, and spelling of his name. According to the Poggendorff Handwörterbuch and old French encyclopedias, Jacques Eugène d'Allonville, Chevalier de Louville, was born 14 July, 1671 at Louville (or

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