SOME RESULTS CONCERNING THE UNION OF FLAT CELLS

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1. Introduction. Many embedding problems reduce to the situation in which one has two cells that are nicely embedded in a euclidean space and would like to conclude that their union is nicely embedded. For example, in order to show that the Annulus Conjecture is true it suffices to show that, if D_1 and D_2 are non-intersecting flat $(n - 1)$-cells in S^n, then $D_1 \cup D_2$ is a flat pair (see §2 for definitions). In [4] it was observed that in order to show that the set of points at which an $(n - 1)$-sphere in S^n, $n > 3$, fails to be locally flat is either empty or uncountable it is sufficient to show that the union of two flat $(n - 1)$-cells, which intersect in an $(n - 2)$-cell that is flat in the boundary of each, is a flat pair.

Let $\alpha(n)$ denote the statement: the Annulus Conjecture is true in dimension n. For $-1 \leq k < m \leq n$, let $\beta(n, m, k)$ denote the statement: if D_1 and D_2 are flat m-cells in $S^n(E^n)$ which intersect in a k-cell that is flat in the boundary of each, then $D_1 \cup D_2$ is a flat pair. Note that in the above examples the conditions are respectively $\beta(n, n - 1, -1)(k = -1)$ means that D_1 and D_2 are disjoint) and $\beta(n, n - 1, n - 2)$, $n > 3$. These are perhaps the most interesting and, unfortunately, the most difficult of the β-statements. In this note we begin an investigation of the β-statements and will be primarily interested in the cases $k = -1$ and $k = 0$.

2. Definitions.

Definition 1. Let S^n be given the triangulation it inherits as the boundary of an $(n + 1)$-simplex, and let K be an m-sphere (m-cell) in S^n. We say that K is flat if there is a homeomorphism h of S^n onto itself such that $h(K)$ is the boundary of an $(m + 1)$-simplex; ($h(K)$ is an (m-simplex). This definition is equivalent to that given in [1], [4], but is better suited for certain proofs given in this paper.

Definition 2. Let D_1 and D_2 be m-cells in $S^n(E^n)$ such that $D_1 \cap D_2 = \text{Bd}\, D_1 \cap \text{Bd}\, D_2$ is a k-cell. We say that $D_1 \cup D_2$ is a flat pair if there is a homeomorphism h of $S^n(E^n)$ onto itself such that $h(D_1)$ and $h(D_2)$ are simplexes.

Definition 3. If S_1 and S_2 are non-intersecting $(n - 1)$-spheres in S^n, then $[S_1, S_2]$ denotes the closure of the component of $S^n - (S_1 \cup S_2)$ whose boundary is $S_1 \cup S_2$. The Annulus Conjecture in dimension n is that, if

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