A FURTHER GENERALIZATION OF THE SCHAUDER FIXED POINT THEOREM

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Introduction. In two preceding papers [1], [2], the writer established some generalizations of the Schauder fixed point theorem [7] in which the existence of fixed points of a compact mapping f in a Banach space is derived from hypotheses on an iterate f^i of f rather than on f itself. It is our purpose in the present paper to strengthen the results of [2] in the following way:

We consider a continuous self-mapping f of an infinite-dimensional compact convex subset C of the Banach space X. If x_0 is a fixed point of f, x_0 is said to be *repulsive* if there exists a neighborhood U of x_0 in C such that for every x in C different from x_0, $f^i(x)$ eventually falls in $C - U$ and stays in $C - U$. The fixed point x_0 of f is said to be *ejective* if x_0 has a neighborhood U in C such that every x in $U - \{x_0\}$ is mapped outside of U by some iterate f^i of f.

In [2], we showed that f always has a fixed point which is not repulsive. In Theorem 1 below, we show that every mapping f has a fixed point which is not ejective.

Since the criterion for ejectivity of a fixed point x_0 of f is local in character, this result answers a question raised in a letter to the writer by G. Stephen Jones in connection with [2] and related to the applications of the results of [1] which have been given in [3], [4], [5], and [8] to the existence of periodic solutions for differential-difference and functional-differential equations.

Section 1. Let C be an infinite-dimensional compact convex subset of the Banach space X, f a continuous mapping of C into C, $F(=F_f)$ the set of fixed points of f.

Definition 1. *If $x_0 \in F$, x_0 is said to be a repulsive fixed point of f if there exists an open neighborhood U of x_0 in C such that for each x in $C - \{x_0\}$, there exists an integer $j(x)$ such that $f^j(x) \in C - U$ for $j \geq j(x)$.*

Definition 2. *If $x_0 \in F$, x_0 is said to be an ejective fixed point of f if there exists an open neighborhood U of x_0 in C such that for any x in $U - \{x_0\}$, there exists a positive integer $k(x)$ such that $f^{k(x)}(x) \in C - U$.*

The fixed point x_0 is said to be strongly ejective if in addition $f(C - U)$ does not contain x_0.

Theorem 1. *f always has at least one fixed point which is not ejective.*

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