A NAGATA'S METRIC WHICH CHARACTERIZES DIMENSION AND ENLARGES DISTANCE

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In this paper we construct a Nagata's metric which characterizes dimension and at the same time enlarges distance (Theorems 1 and 2 below). Two applications of our construction will be given. One is the dimension-preserving completion which allows continuous extensions of countably many continuous mappings whose values are in a complete metric space (Theorems 3 and 4 below). Another is a simple proof for the coincidence of the covering dimension and the large inductive dimension of a metric space (Theorem 5 below).

In the following dim R and Ind R denote the covering dimension of R and the large inductive dimension defined inductively by means of closed sets separating disjoint pairs of closed sets. A metric ρ for a space R is called a *Nagata's n-dimensional metric* if for every $\epsilon > 0$ and every point x of R, $\rho(y_i, S_{\epsilon/2}(x)) < \epsilon$, $i = 1, \cdots, n+2$, imply $\rho(y_i, y_j) < \epsilon$ for some i, j with $i \neq j$, where $S_{\epsilon/2}(x) = \{y : \rho(x, y) < \epsilon/2\}$. (See [5, Theorem 5].)

THEOREM 1. *Let R be a separable metric space with a totally bounded metric ρ_1, such that* $\dim R \leq n(<\infty)$. *Then there exists an equivalent metric ρ such that*

i) *ρ is a Nagata's n-dimensional metric,*
ii) *ρ is totally bounded,*
iii) *$\rho_1(R) \geq \rho(x, y) \geq \rho_1(x, y)$ for any $x, y \in R$.*

Proof. When R has at most one point, the theorem is trivially true. Hence we confine ourselves to the case when R has at least two points. Let ρ_2 be a metric defined by $\rho_2(x, y) = \rho_1(x, y)/\rho_1(R)$. Let $\mathfrak{U}_i = \{U_\alpha : \alpha \in A_i\}$ be a finite open covering of R with mesh $\mathfrak{U}_i < 1/2^{i+2}$ for $i = 0, 2, \cdots$, This mesh is with respect to ρ_2. Let $\mathfrak{V}_1 = \{V_\beta : \beta \in B_1\}$ be a finite open covering of R such that i) $\mathfrak{V}_1 < \mathfrak{U}_1$ (i.e. $\mathfrak{V}_1$ refines $\mathfrak{U}_1$) and ii) order $\overline{\mathfrak{V}}_1 \leq n+1$, where $\overline{\mathfrak{V}}_1 = \{\bar{V} : V \in \mathfrak{V}_1\}$ and order $\overline{\mathfrak{V}}_1$ is the greatest number of elements of $\overline{\mathfrak{V}}_1$ which have a non-vacuous intersection. Set

$$W(x) = \bigcap \{V_\beta : x \in V_\beta \in \mathfrak{V}_1\} - \bigcup \{\bar{V}_\beta : x \notin \bar{V}_\beta \in \overline{\mathfrak{V}}_1\}.$$

Then $\mathfrak{W}_1 = \{W(x) : x \in R\}$ is a finite open covering of R. Moreover, an arbitrary element of $\mathfrak{W}_1$ meets at most $n+1$ elements of $\mathfrak{V}_1$, because $|\{\beta : W(x) \cap V_\beta \neq \emptyset, \beta \in B_1\}| \leq |\{\beta : x \in \bar{V}_\beta, \beta \in B_1\}| \leq$ order $\overline{\mathfrak{V}}_1 \leq n+1$.

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