IDEAL-REPRESENTATIONS OF STONE LATTICES

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In a recent paper "A generalization of Stone's representation theorem for Boolean Algebras", this Journal 30, p. 469–474 (1963) (henceforth cited as [G]), G. Grätzer proves the following:

Theorem. *Every Stone lattice L is isomorphic to a^*-sublattice of the lattice of all ideals of a complete atomic Boolean Algebra, i.e., the Boolean Algebra of all subsets of a set E.*

A Stone lattice is a distributive, pseudo-complemented lattice L with unit e in which $a^* \vee a^{**} = e$ holds for all $a \in L$. Grätzer's proof of this theorem is long and complicated and makes use of deep results from mathematical logic. It is the aim of this note to give a simple and straightforward proof of this theorem, based directly on Birkhoff's classical representation theorem for distributive lattices. Whereas Grätzer's proof gives no hint how to construct the set E, our proof shows that E can essentially be taken to be the set of all dual prime ideals of L. Furthermore, our proof shows that ideals of a very simple type are sufficient to furnish the desired representation. For terminology and basic facts about Stone lattices, and for further references, see [G]. (In [G] an algebraic proof is given for the special case where L has only one atom. We shall show that a similar simple construction can be used in the general case as well.)

The proof. Using Birkhoff's representation theorem for distributive lattices we can assume that our given Stone lattice is a lattice $\mathfrak{L}$ of subsets of a set E such that finite joins and meets are set theoretical unions and intersections and such that the empty set $\varnothing$ and the entire set E are the zero and the unit of $\mathfrak{L}$, respectively. Replacing, if necessary, the elements of E by infinite sets, we can assume furthermore that for any two elements $A, B \in \mathfrak{L}$ with $A \not\subseteq B$ the set theoretical difference $A - B$ is infinite.

For a given $A \in \mathfrak{L}$ define $I_A = \{M \mid M \subseteq A^{**} \text{ and } M - A \text{ is finite}\}$. Obviously I_A is an ideal in the power set $P(E)$ of E. We show that the mapping $A \to I_A$ is an isomorphism between $\mathfrak{L}$ and a*-sublattice of the lattice of all ideals of $P(E)$.

One obviously has ($A, B \in \mathfrak{L}$):

$$\text{(1)} \qquad \text{if } A \subseteq B, \quad \text{then} \quad I_A \subseteq I_B .$$

If $A \not\subseteq B$, then the set $A - B$ is infinite by construction; hence $A \in I_A$ and $A \notin I_B$. It follows:

$$\text{(2)} \qquad \text{if } A \not\subseteq B, \quad \text{then} \quad I_A \not\subseteq I_B .$$

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