THREE THEOREMS ON THE DEGREES OF RECURSIVELY ENUMERABLE SETS

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The natural field of study that arises from post's work [8] concerns what effect (if any) certain restrictions on the complement of a recursively enumerable (r.e.) set have on its degree of unsolvability. We present three results in this field here: first, that there are semicreative sets in all non-zero r.e. degrees; secondly, that there are non-hypersimple simple sets in all non-zero r.e. degrees; and lastly, that there exists a maximal set of degree $\mathbf{0}'$.

All necessary background material may be extracted from the works listed at the end of the paper; in particular, our notation is based mainly on that of [5]. As is usual now, we differ from [5] in calling a set r.e. if there is a recursive predicate Γ such that it is equal to $\hat{x}(\exists y)\Gamma(x, y)$; this allows the null set to be r.e. We use the standard enumeration of the r.e. sets, $R_0, R_1, \cdots$, that is obtained by setting $R_e = \hat{x}(\exists y)T_1(e, x, y)$ for each e; along with this we set $R_e^n = \hat{x}(\exists y)_{y<n}T_1(e, x, y)$ for each e and n. In analogy with the definition of r.e. set, we shall say that a sequence of r.e. sets $(E_0, E_1, \cdots)$ is r.e. if there is a recursive predicate Γ such that $E_n = \hat{x}(\exists y)\Gamma(n, x, y)$ for each n. In particular, a sequence of finite sets $(F_0, F_1, \cdots)$ is called *strongly* r.e. if there is a recursive function γ such that $F_n = \hat{x}(\exists y)(\gamma(n))_y = x + 1$ for all n, where $(z)_i$ is defined for all z to be the $i + 1$-th exponent in the factorization of z. It is known, for example, from [14], that there are r.e. sequences of finite sets which are not strongly r.e. and in fact such sequences are very easy to construct. The reader may also refer to [14] for the definitions of simple sets, hypersimple sets, hyperhypersimple sets and maximal sets. Lastly, a degree of unsolvability is called r.e. if it is the degree of some r.e. set, the degree of the recursive sets is denoted by $\mathbf{0}$ and the highest r.e. degree by $\mathbf{0}'$.

1. Semicreative sets. Prominent among the r.e. sets which are neither recursive nor simple are the semicreative sets, introduced indirectly by Dekker [2] when he discussed what he called semiproductive sets: a set is semicreative when it is r.e. and its complement is semiproductive. Every creative set is semicreative, but it was shown by Shoenfield [12] that there are semicreative sets which are not creative. He defined an intermediate type of set which he called quasicreative, proved all such sets to be of degree $\mathbf{0}'$, and constructed one that could be shown to be not creative. In this section we prove a result that implies the existence of semicreative sets which are not quasicreative by answering in the negative another natural question: Is every semicreative set

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