A BOUNDED AUTOMORPHIC FORM OF DIMENSION ZERO IS CONSTANT

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1. In recent years several proofs have been given of the fundamental theorem that a bounded automorphic function is constant [1], [3], [4]. In this note we present a new proof which is not only simpler than those cited, but also more general in that we actually prove that *any bounded automorphic form of dimension zero is constant.* This result is new, to the best of our knowledge.

In spite of the simplicity of our proof it is somewhat technical. Thus we include (§4) another proof which is both extremely simple and nontechnical, but which covers only the classical case of automorphic *functions.* This proof too is new; we feel that it is easily accessible to the general mathematical reader.

Suppose Γ is a discontinuous function group with domain of existence $\mathfrak{D}$ (cf. [2; 134] for the definitions). By an automorphic form of dimension zero on Γ we mean here a function $f(\tau)$, meromorphic in $\mathfrak{D}$, such that

$$f(V\tau) = v(V)f(\tau), \quad \text{all} \quad V \in \Gamma \quad \text{and} \quad \tau \in \mathfrak{D}, \tag{1}$$

and which tends to a definite limit on approach to a parabolic vertex from within a fundamental region. Here $v(V)$ is independent of τ and $|v(V)| = 1$ for all $V \in \Gamma$. Of course v is a multiplicative character on Γ. A precise statement of our result is

Theorem 1. *Suppose Γ is a discontinuous function group with domain of existence $\mathfrak{D}$. Assume that Γ has a fundamental region $R \subset \mathfrak{D}$ with the property that $\bar{R} \cap \mathrm{Bd}\mathfrak{D}$ either is empty or consists entirely of parabolic vertices. Then any automorphic form $f(\tau)$ of dimension zero on Γ which is bounded in $\mathfrak{D}$ is a constant.* (All sets are considered as subsets of the entire Riemann sphere.)

As a simple corollary to this theorem we give a new proof (§3) of the fact that, on a compact Riemann surface, *an abelian integral of the first kind all of whose periods are real is constant* (cf. [5; 254] for the usual proof).

2. Proof of Theorem 1. In order to simplify matters somewhat we assume that ∞ is not a parabolic vertex of Γ. This is no loss of generality; for if ∞ is a parabolic vertex of Γ, let A be any linear fractional transformation such that $A(\infty)$ is an ordinary point of Γ. Then ∞ is an ordinary point of $A^{-1}\Gamma A$ and we replace Γ by $A^{-1}\Gamma A$, $f(\tau)$ by $f(A\tau)$, and v by the character v^* defined on $A^{-1}\Gamma A$ by

$$v^*(V) = v(AVA^{-1}), \qquad V \in A^{-1}\Gamma A.$$

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