ANOTHER GENERALIZATION OF THE SCHAUDER FIXED POINT THEOREM

By Felix E. Browder

In an earlier paper [1], the writer established some generalizations of the well-known Schauder fixed point theorem [10] in which hypotheses were imposed on f^m, the m-th iterate of a compact self-mapping of a Banach space X, rather than on f itself. In the past several years, these results have found interesting applications in proofs of the existence of periodic solutions of functional-differential equations [2], [4], [5], [11]. In connection with these applications, G. Stephen Jones in a conversation with the writer has raised the question of establishing a sharper form of the Schauder theorem in the following direction:

Let C be an infinite dimensional compact convex subset of the Banach space X, f a continuous self-mapping of C. If x_0 is a fixed point of f, x_0 is said to be repulsive if there exists a neighborhood U of x_0 in C such that for each neighborhood U_1 of x_0, there is an integer $m(U_1)$ such that for $m \geq m(U_1)$ and $x \varepsilon C - U_1$, we have $f^m(x) \varepsilon C - U$. Otherwise x_0 is said to be non-repulsive. The question is the existence of non-repulsive fixed points. The answer to this question is affirmative.

Theorem 1. *f has a non-repulsive fixed point.*

Jones [6] has established a weaker variant of this theorem based on the results of [1] which asserts (in the present writer's terminology) the existence of a fixed point of f which is not locally strongly repulsive, where the fixed point x_0 is said to be locally strongly repulsive if there exist constants $c > 0$, a with $0 < a < 1$, such that for all x in some neighborhood of x_0 in C,

$$||f(x) - x_0|| \geq c \, ||x - x_0||^a.$$

The proof of Theorem 1 which is given below is based upon the application of Lemma 1 of [1] which in turn rests upon the Lefschetz fixed point theorem [8], [9]. The particular form of this application (Lemma 1, below) is close to a lemma recently announced by W. A. Horn [3] in connection with a somewhat different extension of the results of [1]. An essential role in the proof is played by a theorem of Klee [7] (Lemma 2 below) which asserts that all infinite-dimensional compact convex sets in Banach spaces are mutually homeomorphic.

§1 is devoted to the proof of various lemmas to be applied in the proof of Theorem 1. §2 presents that proof and gives extensions in Theorems 2 and 3.

The writer is indebted to G. Stephen Jones for stimulating his interest in

Received April 30, 1964. Supported in part by the Sloan Foundation, National Science Foundation Grants NSF-G19751 and NSF-GP2283, and Army Research Office (DURHAM) Grant ARO(D)-31-124-G455.