HYPERBOLIC CAPACITY AND INTERPOLATING RATIONAL FUNCTIONS

BY J. L. WALSH

The object of this note is twofold: 1) to establish (§§1–3) some results concerning hyperbolic (non-euclidean) capacity of plane point sets, extending theorems due to Tsuji [2] and Pommerenke [1], and 2) to apply (§§4–8) these results in the study of interpolation and approximation by rational functions, including in particular the concept of maximal (greatest geometric degree of) convergence. Fekete himself used his original idea of the transfinite diameter to study interpolation by polynomials. The sequence of theorems developed in the present note is largely the hyperbolic analog of the discussion previously given [3, §§7.1–7.4], due primarily to Fejér, Kalmár, Fekete, and Walsh.

1. **Uniformly distributed points.**

THEOREM 1. *Let* Γ *denote the unit circumference* $|z| = 1$, H *the interior of* Γ, *and let* E *be a closed set in* H *bounded by a finite number of mutually exterior Jordan curves. Let the function* $h(z)$ *be harmonic in* $K = H - E$, *continuous in the corresponding closed region, equal to zero on* Γ *and to unity on the boundary* B *of* E. *For each* n *let the points* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ *and* $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ *be uniformly distributed on* B *and* Γ *respectively with respect to the conjugate of* $h(z)$. *Then we have*

$$\lim_{n\to\infty} \log \left|\frac{(z - \beta_{n1}) \cdots (z - \beta_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}\right|^{1/n} = \frac{2\pi h(z)}{\tau}, \tag{1}$$

uniformly on any closed set in K,

$$\lim_{n\to\infty} \log \left|\frac{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}{(\bar{\alpha}_{n1}z - 1) \cdots (\bar{\alpha}_{nn}z - 1)}\right|^{1/n} = \frac{-2\pi h(z)}{\tau}, \tag{2}$$

uniformly on any closed set in $K_1 = K + \Gamma + K'$, *where* τ *denotes the oscillation on* Γ *of the (any) function conjugate to* $h(z)$, *and* K' *is the inverse in* Γ *of the region* K.

Equation (1) follows at once from the method of proof of [3, §8.7, Theorem 9], as does

$$\lim_{n\to\infty} \log \left|\frac{(z - \beta_{n1}) \cdots (z - \beta_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}\right|^{1/n} = 0 \tag{3}$$

uniformly for z on any closed bounded set in K'. At corresponding points z

Received April 6, 1964. Research sponsored (in part) by the Air Force Office of Scientific Research. Abstract published in Notices of the American Mathematical Society, vol. 11(1964), p. 388.