NOTES ON A COMBINATORIAL THEOREM OF BOHNENBLUST

By Roger H. Farrell

0. Summary. Stated below is a combinatorial result communicated by Bohnenblust to Spitzer a number of years ago. Since that time no proof has appeared. In view of considerable interest at present, it is the purpose of these notes to make available a proof of Bohnenblust's Theorem.

1. Introduction. Throughout, $\Omega = \{1, 2, \cdots, n\}$. A set function $\epsilon(\cdot)$ is a real valued function defined on the subsets of Ω such that if S and T are disjoint subsets of Ω, then

$$\epsilon(S) = \epsilon(S \cup T) \quad \text{or} \quad \epsilon(T) = \epsilon(S \cup T).$$

We mention two examples. Suppose $x_1, \cdots, x_n$ are n real numbers. Let $\theta(\cdot)$ be defined by $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x \leq 0$. For any subset $S = [\delta_1, \cdots, \delta_j]$ define $\epsilon(S) = \theta(x_{\delta_1} + \cdots + x_{\delta_j})$. The function $\epsilon(\cdot)$ defined in this way is a set function. To obtain a different example, for any subset $S = [\delta_1, \cdots, \delta_j]$ define $\eta(S) = \max_{1 \leq i \leq j} x_{\delta_i}$. Then $\eta(\cdot)$ is a set function.

Easily verified are : a set function on Ω can have at most n values; if $S \subset \Omega$, then there is a $j \,\varepsilon\, \Omega$ such that $\epsilon(\{j\}) = \epsilon(S)$.

The sequel will discuss *sequences* $\delta_1, \delta_2, \cdots, \delta_j$, *permutations*

$$\delta = \begin{pmatrix} 1, & 2, & \cdots, n \\ \delta_1 & \delta_2, & \cdots, \delta_n \end{pmatrix},$$

and permutations $(\rho_1, \cdots, \rho_i)$ which are cycles. Since a sequence $\rho_1, \cdots, \rho_i$ uniquely determines the cycle $(\rho_1, \cdots, \rho_i)$ (but not conversely). the notation $[\rho_1, \cdots, \rho_i]$ can be read "the set whose elements are the numbers in the sequence $\rho_1, \cdots, \rho_i$" or "the set whose elements are the numbers in the cycle $(\rho_1, \cdots, \rho_i)$." Throughout we will write $\epsilon[\rho_1, \cdots, \rho_i]$ instead of $\epsilon([\rho_1, \cdots \rho_i])$.

Given a permutation δ and an integer k between one and n there is determined a unique cycle in the cyclic decomposition of δ which contains k. We write $c(\delta, k)$ for this cycle. In the sequel we write G for the collection of $n!$ permutations of the set $\Omega = \{1, 2, \cdots, n\}$. We define functions f and g on the Cartesian product $G \times \Omega$ as follows. Let $\delta \,\varepsilon\, G$ and

$$\delta = \begin{bmatrix} 1, \cdots, n \\ \delta_1, \cdots, \delta_n \end{bmatrix}.$$

Then,

$$f(\delta, k) = \epsilon[\delta_1, \cdots, k];$$

Received March 9, 1964.