MINIMAL PROJECTIVE EXTENSIONS OF COMPACT SPACES

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A compact space E is called *projective* if for each mapping ψ of E into a compact space X, and each continuous mapping τ of a compact space Y *onto* X, there is a continuous mapping ϕ of E into Y such that $\psi = \tau \circ \phi$. Gleason proved in [1] that a compact space E is projective if and only if it is extremally disconnected. (A topological space E is *extremally disconnected* if the closure of each of its open sets is open. It is well known that E is extremally disconnected if and only if the Boolean algebra of open and closed subsets of E is complete.) Gleason showed, moreover, that for each compact space X, there is a unique compact extremally disconnected space $\mathfrak{R}(X)$, and a continuous mapping π_X of $\mathfrak{R}(X)$ onto X such that no proper closed subspace of $\mathfrak{R}(X)$ is mapped by π_X onto X. (An alternate development of Gleason's results is given by Rainwater in [2].) We call $\mathfrak{R}(X)$ the *minimal projective extension* of X; it can be described as follows.

Let $R(X)$ denote the family of regular closed subsets of X. (A closed subset of X is called *regular* if it is the closure of its interior.) Then $R(X)$ is a complete Boolean algebra if we define for α, β in $R(X)$

$$\alpha \vee \beta = \alpha \cup \beta; \alpha \wedge \beta = \text{cl int } (\alpha \cap \beta).$$

Note that the Boolean complement α^* of α is given by

$$\alpha^* = \text{cl } (X \sim \alpha).$$

The space $\mathfrak{R}(X)$ is the Stone space of $R(X)$. That is, the points of $\mathfrak{R}(X)$ are the prime ideals of $R(X)$, and a base for the topology of $\mathfrak{R}(X)$ is the family of sets $\{P \in \mathfrak{R}(X) : \alpha \notin P\}$, $\alpha \in R(X)$.

The mapping π_X is defined by letting $\pi_X(P) = \bigcap \{\alpha \in R(X) : \alpha \notin P\}$ for each $P \in \mathfrak{R}(X)$.

1. Lemma. *The mapping $\alpha \to \pi_X^{-1}(\alpha)$ is an isomorphism of $R(X)$ onto the Boolean algebra of open and closed subsets of $\mathfrak{R}(X)$.*

From Gleason's theorems we deduce quickly the following induced mapping theorem which motivates this paper.

2. Theorem. *Let τ be a continuous mapping of a compact space Y onto X. Then there exists a continuous mapping $\bar{\tau}$ of $\mathfrak{R}(Y)$ onto $\mathfrak{R}(X)$ such that $\tau \circ \pi_Y = \pi_X \circ \bar{\tau}$. Thus the following diagram is commutative.*

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