A NOTE ON THE AVERAGE ORDER OF NUMBER-THEORETIC ERROR TERMS

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Let $\tau(n) = \sum_{d|n} 1$ and as usual, $\Delta(x) = \sum_{n \le x} \tau(n) - x \log x - (2\gamma - 1)x$ where γ is Euler's constant. Chowla and Walum [2] have used the result

$$\sum_{n \le x} \Delta(n) = \tfrac{1}{2} x \log x + (\gamma - \tfrac{1}{4})x + 0(x^{\frac{3}{4}}),$$

attributing it to Voronoï in whose work of 1904 (unavailable to the author) it is implicit.

In this note we observe that this and many similar formulas may be readily derived from the following elementary lemma about such averages of error terms.

Lemma. *Let $f(n)$ be a function of a positive integral variable, and suppose* (1) $\sum_{n \le x} f(n) = g(x) + E(x)$, *where $g(x)$ is twice continuously differentiable. Then*

$$(2) \qquad \sum_{n \le x} E(n) = \tfrac{1}{2} g(x) + (1 - \{x\})E(x) + \int_1^x E(t)\, dt + O(|g'(x)|) + O(1),$$

where $\{x\} = x - [x]$ indicates the fractional part of x.

Proof.

$$\begin{aligned}
\sum_{n \le x} E(n) &= \sum_{n \le x} \sum_{m \le n} f(m) - \sum_{n \le x} g(n) = \sum_{n \le x} ([x] - n + 1) f(n) \\
&\quad - \int_1^x g(t)\, dt - \int_1^x (\{t\} - \tfrac{1}{2}) g'(t)\, dt + (\{x\} - \tfrac{1}{2}) g(x) + O(1) \\
&= (x + \tfrac{1}{2}) g(x) + ([x] + 1) E(x) - \sum_{n \le x} n f(n) - \int_1^x g(t)\, dt \\
&\quad - \int_1^x (\{t\} - \tfrac{1}{2}) g'(t)\, dt + O(1) = \tfrac{1}{2} g(x) + (1 - \{x\}) E(x) \\
&\quad + \int_1^x E(t)\, dt - O(|g'(x)|) + \int_1^x O(1) g''(t)\, dt + O(1) \\
&= \tfrac{1}{2} g(x) + (1 - \{x\}) E(x) + \int_1^x E(t)\, dt + O(|g'(x)|) + O(1),
\end{aligned}$$

by the Euler-Maclaurin summation formula applied to $\Sigma g(n)$ and partial summation applied to $\Sigma n f(n)$.

Clearly by assuming more stringent differentiability conditions on $g(x)$, one can obtain a more accurate estimate of the error by using further terms of the Euler-Maclaurin formula, but the above is sufficient for the purposes of this note.

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