EXTREME POINTS IN FUNCTION ALGEBRAS

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1. Let X be a nonempty compact Hausdorff space, and denote by $C(X)$ $[C_r(X)]$ the space of all complex [real]-valued continuous functions on X, with supremum norm. It is well known that a function f in either of these spaces is an exteme point of the unit ball $U(= \{f : \| f \| \leq 1\})$ if and only if $|f(x)| = 1$ for each x in X. The question naturally arises as to whether U is the closed convex hull of its extreme points. In the real case, the answer (which seems to be widely known) has been provided by Bade [1] (see, also, Goodner [6]):

The unit ball of $C_r(X)$ is the closed convex hull of its extreme points if and only if X is totally disconnected.

It is less widely known that in the complex case, U is *always* the closed convex hull of its extreme points (Theorem 1). The present paper is concerned with the question of what happens in the case of a *subalgebra A* of $C(X)$. (The analogous question for *subspaces* of $C(X)$ is, of course, too general, since every Banach space is such an object, for a suitable X.) Our attention will be restricted to *function algebras A*; by this, we mean that A is closed, contains the constant functions and separates points of X.

Considerable use will be made of the following characterization of the set ext U of extreme points of the unit ball U of A [10]:

If $g \in U$, then g is not in ext U *if and only if there exists h in A, $h \neq 0$, such that $|g(x)| + |h(x)| \leq 1$ for each x in X.* This characterization makes it easy to construct examples of algebras A in which U is not the closed convex hull of ext U. On the other hand, U *is* the closed convex hull of its extreme points for a number of well-known function algebras A. One of these is the *disc algebra* $\mathcal{A}$ of all complex functions f which are analytic in $|z| < 1$ and continuous in $|z| \leq 1$. This algebra is of particular interest because of the following well-known result: Suppose that A is a function algebra and that $g \in U$. If $f \in \mathcal{A}$, then the composition $f \circ g$ is in A, and if $\| f \| \leq 1$, then $f \circ g \in U$. It was shown in [10] that if $g \in$ ext U, then so is g^n, $n = 1, 2, \cdots$. Thus, one can generate exteme points by starting with such a g and considering compositions of the form $f \circ g$, where $f(z) = z^n$. In §5, we characterize the functions f from the unit ball of $\mathcal{A}$ for which the map $g \rightarrow f \circ g$ preserves extreme points; they turn out to be the "inner functions" in $\mathcal{A}$, i.e., the finite Blaschke products.

As we have noted above, the unit ball of $\mathcal{A}$ is the closed convex hull of its extreme points; more generally, we show in §4 that this is true of a class of algebras which are "somewhat like" $\mathcal{A}$. [This class (which contains H^∞) consists of certain logmodular algebras.] In fact, we prove somewhat more, in that we

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