AN INEQUALITY FOR THE RIEMANN ZETA FUNCTION

By Robert Spira

As a result of extended computations of the Riemann zeta function and of study of the graphs, the following theorem has been conjectured and proved: Let $\zeta(1 - s) = g(s)\zeta(s)$, where $g(s) = (2\pi)^{-s}\, 2 \cos \pi s/2)\Gamma(s)$, (Titchmarsh [2, Chapter 2]).

Theorem 1. *For* $t \geq 10$, $\frac{1}{2} < \sigma < 1$, $|g(s)| > 1$.

The $t = 10$ value is chosen for convenience in the estimates. This property fails for t around 2π, as can be observed in the Jahnke and Emde [1] tables. It is clear that $|g(\frac{1}{2} + it)| = 1$, so that the theorem will be proved provided it is shown that $\partial|g(s)|/\partial\sigma > 0$. Starting the proof, we set $g(s) = UV\Gamma(s)$, where

$$(1) \qquad U = (2\pi)^{-s}, \qquad V = 2 \cos \frac{\pi}{2} s.$$

Now

$$(2) \qquad |V| = 2\left[\cos^2 \frac{\pi}{2}\sigma + \sinh^2 \frac{\pi t}{2}\right]^{\frac{1}{2}} \geq 2 \sinh \frac{\pi t}{2}$$

so that $|V|$ decreases as σ increases, as does $|U| = (2\pi)^{-\sigma}$, leaving the whole burden of the increase upon the Γ-function. Differentiating, we obtain

$$(3) \qquad \frac{\partial}{\partial\sigma}|U| = -|U| \log 2\pi, \qquad \frac{\partial}{\partial\sigma}|V| = -(\pi \sin(\pi\sigma))/\,|V|.$$

To estimate the Γ-function, we use the Stirling formula (de Bruijn [3]):

$$(4) \qquad \log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + \sum_{k=1}^{m} s^{1-2k}(2k)^{-1}(2k - 1)^{-1}B_{2k} + R_m$$

where

$$(5) \qquad R_m = -\int_0^\infty (s + x)^{-2m}(2m)^{-1}B_{2m}(x - [x])\, dx.$$

For the estimation of R_m, and for other estimates, we have:

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