A STRUCTURAL PROPERTY OF CERTAIN LOCALLY COMPACT ABELIAN GROUPS

By Edwin Hewitt

Let a be an integer > 1. A locally compact group G is said to be a-rich if for every neighborhood U of the identity e in G, the set $\{x^a : x \in U\}$ has positive [left] Haar measure. If G is not a-rich, we say that it is a-meager. The class of 2-rich G was introduced by Devinatz and Nussbaum in [1], and was used in studying real characters. In this note, we classify all a-rich locally compact Abelian groups.

Henceforward let G be a locally compact Abelian group. Let σ_a be the continuous endomorphism $x \to x^a$ of G. Write $G = R^n \times G_0$, where n is a non-negative integer and G_0 is a locally compact Abelian group containing a compact open subgroup H (see [2, (24.30)]). Since $R^n \times H$ is an open subgroup of G and R^n is a divisible group, it is clear that G is a-rich if and only if H is a-rich.

Let λ be normalized Haar measure on H. Suppose that H is a-meager, and that $\lambda(\sigma_a(U^-)) = 0$ for some neighborhood U of e in H. Since H is compact, there is a finite subset $\{x_j\}_{j=1}^m$ of H such that

$$H = \bigcup_{j=1}^m x_j U = \bigcup_{j=1}^m x_j U^- .$$

Thus we have

$$\lambda(\sigma_a(H)) = \lambda\left(\bigcup_{j=1}^m \sigma_a(x_j)\sigma_a(U^-)\right) \leqq \sum_{j=1}^m \lambda(\sigma_a(x_j)\sigma_a(U^-))$$
$$= 0.$$

Hence H is a-rich if and only if the compact subgroup $\sigma_a(H)$ of H has positive Haar measure. This obviously occurs if and only if $H/\sigma_a(H)$ is finite.

The assertion that $H/\sigma_a(H)$ is finite is purely algebraic and can be described in terms of the algebraic structure of H, which is completely known (see [2, (25.25)]). The group H is isomorphic with

$$\text{(1)} \qquad \mathbf{P}_{p \in P} [\Delta_p^{\mathfrak{a}_p} \times \mathbf{P}_{i \in I_p} Z(p^{r_i})] \times \{\mathbf{P}^*_{p \in P} Z(p^\infty)^{\mathfrak{b}_p *} \times Q^{\mathfrak{n} *}\}.$$

Here Q is the additive rationals, P is the set of prime positive integers, $Z(p^\infty)$ is the p^∞-group, $Z(p^r)$ is the cyclic group of order p^r [r a positive integer], and Δ_p is the p-adic integers. The symbol $\mathbf{P}^*$ is a weak direct product and $\mathbf{P}$ is a complete direct product. The cardinal numbers $\mathfrak{b}_p$ and $\mathfrak{n}$ are subject to certain conditions of no present interest. The cardinal numbers $\mathfrak{a}_p$ are arbitrary. The

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