THE DIMENSION OF $A[x]$

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Introduction. Let R be a commutative ring with identity. By dim R is meant the supremum of all n for which there is a chain

$$P_0 \subset P_1 \cdots \subset P_n$$

of prime ideals of R ($\subset$ denotes strict inclusion). The height of a prime ideal P of $R(h(P))$ is defined to be dim R_P, and the depth of a prime ideal P of $R(d(P))$ is defined to be dim R/P. The present article deals with relations between dim $A[x]$ and dim A, where A is a noetherian domain of finite dimension and $A[x]$ is a domain which is a simple ring extension of A.

The following two results will be used repeatedly.

Theorem 1. *A prime ideal P of a noetherian ring R has height $\leq m$ if and only if P is an isolated prime ideal of an ideal of R generated by m elements* [1, Chapter 4, Theorems 30 *and* 31].

Theorem 2. *Let R be a ring, P and Q two prime ideals of R such that $P \subset Q$, R' an overring of R integral over R, and P' a prime ideal of R lying over P. Then there exists a prime ideal Q' of R' containing P' and lying over Q* [1, Chapter 5, Corollary to Theorem 3].

Proposition 3. *If $A[X]$ is the polynomial ring over A, then* dim $A[X]$ = dim $A + 1$.

Remark. This proposition is well known, and what follows is probably the shortest proof.

Proof. If dim $A = n$, then there is a chain of prime ideals

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

of A. We get a chain of prime ideals

$$P_0A[X] \subset P_1A[X] \subset \cdots \subset P_nA[X] \subset P_nA[X] + (X)$$

of $A[X]$ of greater length yielding dim $A[X] \geq n + 1$.

Let M be any maximal ideal of $A[X]$. Set $P = A \cap M$, and let $S = A - P$. The natural isomorphism of $A_S[X]$ onto $A[X]_S$ induces a homomorphism f:

$$A_S[X]/PA_S[X] \rightarrow A[X]_S/MA[X]_S .$$

But $A_S[X]/PA_S[X] \simeq (A_S/PA_S)[X]$, and A_S/PA_S is a field (the quotient field of A/P). Since $MA[X]_S$ is maximal in $A[X]_S$, the kernel of f must be generated

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