NONCONTINUABLE ANALYTIC FUNCTIONS

By F. W. Carroll and J. H. B. Kemperman

1. Introduction. The object of the present paper is to present new classes of power series

$$(1.1) \qquad f(z) = \sum_{n=0}^{\infty} f_n z^n, \qquad \overline{\lim}\, |f_n|^{1/n} = 1,$$

having $|z| = 1$ as a natural boundary; some of these were already announced in [3]. Our results are closely related to the well-known observation of Hecke [7], see also [10; 106] that (1.1) with $f_n = [n\alpha]$ has $|z| = 1$ as a natural boundary whenever the real constant α is irrational. This result was subsequently generalized by Salem [19], Newman [17], Mordell [15], [16], Schwarz [20] and Meijer [14].

The results of §2 are simply obtained by using Hecke's method in conjunction with the Hadamard multiplication theorem. However, since Hecke's method is bound to Riemann integrability, these results are correspondingly limited. In §4 we consider (1.1) with $f_n = f_n(\alpha)$ as given (Lebesgue) integrable functions, and are interested in conditions for (1.1) to have $|z| = 1$ as a natural boundary for almost all α. For instance:

Theorem 1.1. *Let $\varphi(x)$ be any integrable function on $0 \leq x < 1$. Then the power series*

$$(1.2) \qquad f_\alpha(z) = \sum_{n=0}^{\infty} \varphi(((n\alpha)))z^n$$

has for almost all real α a radius of convergence ≥ 1.

Moreover, $f_\alpha(z)$ has for almost all α the circle $|z| = 1$ as a natural boundary if and only if $\varphi(x)$ is not equivalent to a trigonometric polynomial $\sum c_p e^{2\pi i p x}$ of period 1.

Here, $((x)) = x - [x]$ denotes the fractional part of the real number x. By "equivalent" we mean "equal almost everywhere".

Theorem 1.1 and its generalization Theorem 4.1 are based upon the auxiliary Theorem 3.1. The latter is given in a form more general than needed in the proof of Theorem 4.1, since it seems to be of independent interest.

Finally, in §5 we derive some new results from the classical Carlson-Pólya theorem on power series with integral coefficients and its recent generalization by Popken [18]. In this direction, Meijer [14] already proved that (1.1) has $|z| = 1$ as a natural boundary when $f_n = P([\alpha n^k])$ with $P(u)$ as a polynomial of degree ≥ 1 with algebraic coefficients, α as an irrational number, k as a positive integer.

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