INEQUALITIES FOR MAPPINGS ON SPACES OF SKEW-SYMMETRIC TENSORS

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1. Introduction. The analysis of certain operators on appropriate symmetry classes of tensors has proved to be an effective method for obtaining inequalities for matrix functions. In [4] this technique was used to obtain a partial resolution of the van der Waerden conjecture concerning the permanent of a doubly stochastic matrix. Again in [3] an analogue for permanents of the Hadamard determinant theorem was obtained by analyzing the eigenvalues and eigenvectors of a positive definite hermitian operator on the symmetry class of completely symmetric tensors. In this paper we generalize an inequality recently published by Thompson [5] by obtaining the eigenvalues and eigenvectors of a linear operator connecting two appropriate Grassmann spaces.

2. Statement of the inequality. Let A be an mk-square complex hermitian positive definite matrix. Suppose A is partitioned into m^2 k-square submatrices A_{ij} : i.e. A_{ij} is the submatrix of A lying in rows $k(i-1)+1, \cdots, ki$ and columns $k(j-1)+1, \cdots, kj$, $i, j = 1, \cdots, m$. For $t = 1, \cdots, m$ let B_t be the t-square matrix whose i, j entry is $\det A_{ij}$, $i, j = 1, \cdots, t$, and let A_t be the principal submatrix of A lying in rows and columns $1, \cdots, tk$ of A. We prove the following result.

THEOREM. *For each* $t = 2, \cdots, m$,

$$\frac{\det B_{t-1}}{\det A_{t-1}} \leq \frac{\det B_t}{\det A_t}, \qquad \frac{\det B_1}{\det A_1} = 1. \tag{1}$$

Equality holds in (1) *if and only if* $A_t = A_{t-1} \dotplus A_{tt}$, *where* $\dotplus$ *indicates direct sum.*

In [1] Everitt proved that $\det B_2/\det A_2 \geq 1$, and in [5] Thompson proved the generalization: $\det B_t/\det A_t \geq 1$.

3. Proof of the theorem. Let U be an n-dimensional unitary space, $n \geq mk$, with inner-product $(,)$. As usual $\overset{p}{\wedge} U$, $1 \leq p \leq n$, will denote the p-th Grassmann space associated with U, i.e. the symmetry class of all skew-symmetric p-contravariant tensors defined on U, [2]. The inner product in U naturally induces an inner product $(,)_p$ in $\overset{p}{\wedge} U$ whose value on a pair of pure vectors

Received August 16, 1963. The research of both authors was supported by the National Science Foundation under grant G. P. 1085.