DIFFERENTIABLY SIMPLE RINGS OF PRIME CHARACTERISTIC

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1. Introduction. Let D be a set of derivations on a commutative ring R with identity 1; an ideal I of R is called a *D-ideal* if $dI \subset I$ for all d in D; R is called a *differentiably simple ring* with respect to D, or a *D-simple ring*, if the only D-ideals of R are $\{0\}$ and R. As we shall see, if R is D-simple, then the identity 1 generates a field in R; the characteristic of this field is defined to be the characteristic of R. For the characteristic zero case, E. Posner [3] [4] proved that, among other things, R is always an integral domain, and under some finiteness condition, integrally closed in its quotient field. When the characteristic is $p > 0$, L. Harper [2; Theorem 1] determined the structure of R in the case that R is a finite dimensional algebra over an algebraically closed field C: it is of the form

$$C[t_1, \cdots, t_n]/(t_1^p, \cdots, t_n^p).$$

(It should be pointed out that Harper's theorem can also be deduced from some work of R. Ree on simple Lie algebras; see [5; Theorem 3.5] and [6; Corollary 2.4].)

In this paper we concern ourselves exclusively with the case where the characteristic is equal to $p > 0$. First we observe that R is always a local ring, its radical N is a nil ideal, and R admits a field of representatives F, i.e. $R = F + N$ (direct sum as additive subgroups). Then with the hypothesis that either N/N^2 or D^*/ND^* is of finite dimension over F (where D^* is the restricted derivation Lie algebra on R generated by D——by a *derivation Lie algebra G on R* we mean an R-module consisting of derivations which is a Lie algebra under the usual operations for derivations, G is *restricted* if $d^p \in G$ for any d in G) and with the help of Harper's technique, we show that R is already of the form

$$F[t_1, \cdots, t_n]/(t_1^p, \cdots, t_n^p),$$
$$n = \text{dimension of } N/N^2 \text{ over } F.$$

We conclude this paper with some remarks on derivations of a differentiably simple ring.

2. Structure theorems.

2.1. Theorem. *Let R be a D-simple ring. Then R is a local ring, and $x^p = 0$ for all x in N. Moreover, R contains a field F isomorphic to R/N under the canonical projection $\pi : R \to R/N$, and F can be assumed to contain the field $K = \{x : dx = 0 \text{ for all } d \text{ in } D\}$.*

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