RELATIVE GROUP EXTENSIONS AND KERNELS

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1. Introduction. The concept of a relative group extension was introduced by Hochschild in [4] to obtain an interpretation of the two-dimensional relative cohomology group $H^2(G, H; M)$ of a group G and a subgroup H with coefficients in a G-module M. Such an extension is a triple (E, φ, τ_H) consisting of a group E which contains M, an epimorphism $\varphi : E \to G$ with kernel M, and a monomorphism $\tau_H : H \to E$ such that $\varphi \circ \tau_H$ is the identity on H. The G-module structure on M must of course be that induced by φ, and it is further required that τ_H be extendible to a *map* $\tau : G \to E$ such that $\varphi \circ \tau$ is the identity on G, and which satisfies $\tau(gh) = \tau(g)\tau(h)$, $\tau(hg) = \tau(h)\tau(g)$ whenever $g \in G$, $h \in H$. If we display such an extension as an exact sequence

$$(1) \to M \to E \xrightarrow{\varphi} G \to (1), \tag{1.1}$$

and set $E_H = \varphi^{-1}(H)$, and $\varphi_H = \varphi \mid H$, then the above definition asserts that the sequence

$$(1) \to M \to E_H \xrightarrow{\varphi_H} H \to (1) \tag{1.2}$$

obtained by restriction of (1.1) to H is split exact with τ_H as splitting monomorphism. With the obvious definition of equivalence, the set of equivalence classes of such extensions forms an abelian group under Baer multiplication which is naturally isomorphic to the relative cohomology group $H^2(G, H; M) = \mathrm{Ext}^2_{(Z(G), Z(H))}(Z, M)$ of the Adamson-Hochschild theory (c.f. Hochschild [4]).

This paper continues the study of (G, H)-extensions by considering the case in which the kernel of the epimorphism φ is an arbitrary group K. An analysis of such extensions leads to the concept of a (G, H)-kernel, and then to the group of such kernels under the relation of "similarity", a generalization of the relation introduced by Eilenberg and MacLane [3] to study the non-relative case. By focusing attention upon those kernels which have a fixed G-module M as center we find that each similarity class of (G, H)-kernels determines a unique element of $H^3(G, H; M)$. This yields an interpretation of $H^3(G, H; M)$ as the group of similarity classes of (G, H)-kernels with center M, and to a characterization of "extendible" kernels as those whose associated cohomology class is trivial. Finally, it is shown that the group of equivalence classes of (G, H)-extensions with abelian kernel M can be made to operate as a simply transitive transformation group on the set of equivalence classes of

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