A STRUCTURE OF THE RAYLEIGH POLYNOMIAL

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The Rayleigh polynomial $\phi_{2n}(\nu)$ has been defined [2], [3] in the following manner: Let $J_\nu(z)$ be the Bessel function of the first kind, and let $j_{\nu,m}$, $m=1, 2, \cdots$, be the zeros of $z^{-\nu}J_\nu(z)$, $|\text{Re}\,(j_{\nu,m})| \leq |\text{Re}\,(j_{\nu,m+1})|$, then

$$(1) \qquad \phi_{2n}(\nu) = 4^n \prod_{k=1}^{n} (\nu + k)^{[n/k]}\sigma_{2n}(\nu),$$

where

$$(2) \qquad \sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n}, \qquad n = 1, 2, \cdots,$$

and $[x]$ is the greatest integer $\leq x$.

The symmetric function $\sigma_{2n}(\nu)$ is called [1] the Rayleigh function of order $2n$, and has been the subject of a number of investigations by Cayley, Watson, Forsyth and others [4; 502]. It is obvious from (1) that any structure of $\sigma_{2n}(\nu)$ is closely related with that of $\phi_{2n}(\nu)$. However, no simple structure of the Rayleigh polynomial $\phi_{2n}(\nu)$ is known so far. It has been shown [2] that $\phi_{2n}(\nu)$ is a polynomial with positive integral coefficients, that its degree is $1 - 2n + \sum_{k=1}^{n} [n/k]$ and that all of its real roots lie in the interval $(-n, -2)$. Consequently, $\phi_{2n}(\nu)$ may be written as

$$(3) \qquad \phi_{2n}(\nu) = \sum_{k=1}^{d} a_{n,k}\nu^k, \qquad d = 1 - 2n + \sum_{s=1}^{n} \left[\frac{n}{s}\right].$$

The object of this paper is to give a structure of the polynomial $\phi_{2n}(\nu)$.

Consider the positive integers s, k and n such that $s \leq n$, $k \leq n$. And let

$$(4) \qquad \epsilon(s, k, n) \equiv \left[\frac{n}{s}\right] - \left[\frac{k}{s}\right] - \left[\frac{n-k}{s}\right].$$

It is seen that the value of $\epsilon(s, k, n)$ is either 0 or 1. Let $R_m(n)$ be the smallest non-negative remainder when n is divided by m. That is,

$$(5) \qquad n - R_m(n) \equiv 0 \pmod{m}, \qquad 0 \leq R_m(n) < m.$$

(6) Lemma. *$R_s(n) < R_s(k)$ if and only if $\epsilon(s, k, n) = 1$.*

Proof. Let $n = as + R_s(n)$
$k = bs + R_s(k)$, where a, b are integers. Then

$$\epsilon(s, k, n) = \left[\frac{n}{s}\right] - \left[\frac{k}{s}\right] - \left[\frac{n-k}{s}\right]$$

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