TAMING COMPLEXES IN HYPERPLANES

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1. Introduction. In this paper we investigate conditions that suffice for an imbedding of a complex in a Euclidean space to be tame (cf. §§2, 7 for definitions). We also consider the more general question of when two imbeddings are equivalent. We show, for example, with a dimension restriction that if the image under an imbedding of a k-complex is contained in a hyperplane of codimension k, then the imbedding is tame. More precisely we prove:

THEOREM 1.1. *Let K be a finite k-dimensional complex and h an imbedding of K into an n-plane E^n in E^{n+k}, where $k + 2 \leq n$. Let ϵ be any positive number. Then there exists an isotopy $g_t (t \varepsilon I)$ of E^{n+k} onto itself such that*

1) *g_0 is the identity,*
2) *$g_1 h$ is piecewise linear,*
3) *g_t is the identity outside an ϵ-neighborhood of $h(K)$ for each t in I,*
4) *each point of E^{n+k} moves along a polygonal path under $g_t(t \varepsilon I)$ having length less than ϵ.*

The case $k = 1$ of this theorem was substantially proved in [1].

In Corollary 5.6 we establish the corresponding result showing any two such imbeddings into hyperplanes are equivalent (and under an economical isotopy). In doing this we give a new proof (Theorem 5.5) of a result of Gugenheim [5] which leads to a sharpened form of his Theorem 5.

The first result of the type we prove in Corollary 5.6 is due to Klee [9] who showed that if a compact set K can be imbedded in E^l, then any two imbeddings of K into hyperplanes of codimension l are equivalent. In Theorem 6.2 we improve Klee's result for certain sets, not necessarily complexes, which leads to stronger results in one sense (we can drop the requirement that $k + 2 \leq n$) than Corollary 5.6 for 2-manifolds (Corollary 6.3) and k-spheres with handles (Corollary 6.4). In [10] Stallings establishes the equivalence of locally flat imbeddings of k-spheres in E^n, $n \geq k + 3$, which, when combined with Klee's result, would provide an alternate proof of Corollary 6.4 in the special case of a k-sphere.

Homma [6] and Gluck [3] have shown that the tameness of an imbedding in a combinatorial manifold is guaranteed if the imbedding is well-behaved in a neighborhood of the image. Gluck, for example, has shown that a locally tame imbedding is tame in the dimension range we are considering. (The authors have just recently learned that C. A. Greathouse [4] has also obtained

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