SOME EXPLICIT INEQUALITIES FOR UNIFORMLY ELLIPTIC OPERATORS

By L. E. Payne

1. Introduction. In a series of recent papers, Fichera [1], [2], [3], has made use of L_p inequalities to derive a number of maximum norm inequalities. We shall be concerned in this note with one of the types of problems considered in [1]. By a generalization of Fichera's method it is possible to treat a case not considered in his papers.

Following Fichera's notation, we let D be a connected domain (open set) with boundary Σ in R_n. Let $E(u)$ denote the linear differential operator

$$E(u) = (a^{ij}u_{,i})_{,j} \tag{1.1}$$

where a^{ij} is a positive definite symmetric matrix. In (1.1) the repeated indices i and j are to be summed from 1 to n and the comma denotes differentiation.

We assume throughout that E is a uniformly elliptic operator even though the results carry over rather easily to certain classes of elliptic-parabolic operators. We assume further for simplicity that the a^{ij} have continuous first derivatives. From the results obtained, it is clear that by approximating in terms of smooth functions, it would be possible to obtain the same result with less stringent smoothness requirements on the a^{ij}.

We derive in the next section a number of L_p-inequalities from which follow maximum norm inequalities for u in terms of the Dirichlet data of u. Our results would carry over with little difficulty to the more general operator

$$\tilde{E}(u) = E(u) + cu, \qquad c \leq 0. \tag{1.2}$$

2. A priori inequalities. Let h be any solution of $E(h) = 0$ in D. We establish then the following theorem:

Theorem I. *If φ is any piecewise C^2 function in D which satisfies*

$$\begin{aligned} E(\varphi) &\leq 0 \quad \text{in} \quad D \\ \varphi &\geq 0 \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.1}$$

then for any k in the interval $0 \leq k \leq 1$ it follows that

$$\max_D \left| \frac{h}{\varphi^k} \right| \leq \max_\Sigma \left| \frac{h}{\varphi^k} \right|. \tag{2.2}$$

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