THE NUMERICAL RANGE OF A NORMAL OPERATOR

By S. K. Berberian

One of the most striking properties of a normal operator is that the closure of its numerical range coincides with the convex hull of its spectrum [7, Theorem 8.14]. The object of this note is to give a new proof of this result. Our proof is quite algebraic; in particular, we shall rely on the "functional representation" formulation of the spectral theorem, rather than the "spectral integral" formulation (see Lemma 5). In the preliminary lemmas there are a few wisps of greater generality, fostering the hope that the theorem may have a natural generalization to a suitable class of not necessarily normal operators. In particular, we conjecture that the conclusion of the theorem holds for a hyponormal operator [1].

Our notation and terminology conforms with that of [4]. In particular, *operator* means continuous linear mapping in Hilbert space, and scalar products are denoted (x, y). The *spectrum* of an operator T, denoted $\Lambda(T)$, is the set of all complex numbers λ such that $T - \lambda I$ is not invertible [4, §21]; its convex hull is a compact convex set [3; 85, Exercise 2]. The *numerical range* of an operator T, denoted $W(T)$, is the set

$$W(T) = \{(Tx, x) : ||x|| = 1\}$$

[7, Definition 4.3]. We write $\langle T \rangle$ for the closure of $W(T)$, and call it the *closed numerical range* of T; thus, $\lambda \varepsilon \langle T \rangle$ if and only if there is a sequence of unit vectors x_n such that $(Tx_n , x_n) \rightarrow \lambda$. The numerical range of an operator is a (bounded) convex set, by the Toeplitz-Hausdorff theorem [7, Theorem 4.7]; consequently, the closed numerical range of an operator is a compact convex set. For completeness we include a proof of the following well-known result:

Lemma 1. *The spectrum of any operator is contained in its closed numerical range.*

Proof. Suppose first that μ is a boundary point of $\Lambda(T)$. Since μ is an approximate proper value for T [6, Theorem 66-B], there is a sequence of unit vectors x_n such that

$$||Tx_n - \mu x_n|| \rightarrow 0,$$

and therefore $\mu = \lim (Tx_n , x_n) \varepsilon \langle T \rangle$. If now λ is an arbitrary point of $\Lambda(T)$, let L be any line through λ, and let A be the intersection of L with $\Lambda(T)$. If μ_1 and μ_2 are the endpoints of the smallest closed segment containing A, it is clear that the μ_i are boundary points of $\Lambda(T)$, and so $\mu_i \varepsilon \langle T \rangle$; since $\langle T \rangle$ is convex, it contains the segment joining the μ_i , and in particular $\lambda \varepsilon \langle T \rangle$. Incidentally, when T is normal, Lemma 1 follows instantly from [4, Theorem 31.2], without appeal to the Toeplitz-Hausdorff theorem.

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