GENERALIZED HILBERT KERNELS

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1. Introduction. A function $k(x, y)$ is called a Fourier kernel if for some functions $f(x)$, $g(x)$:

$$g(x) = \int_0^\infty k(xy)f(y)\,dy$$

implies

$$f(x) = \int_0^\infty k(xy)g(y)\,dy. \tag{1}$$

It is known that its Mellin transform $k^\star(s)$ satisfies the functional relation $k^\star(s)\,k^\star(s-1) = 1$, [5; 212–213]. Using the function $k^\star(s) = \cot s\Pi/2$ the kernel $k(x) = 2/\Pi \cdot 1/1 - x^2$ is obtained.

From the reciprocal formulas (1) with this kernel it is possible to derive the reciprocity for Hilbert transforms,

$$g(x) = \frac{1}{\Pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt, \qquad f(x) = \frac{-1}{\Pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{g(t)}{x-t}\cdot dt, \qquad [5; 219].$$

This suggests that similar results might be obtained by taking $k^\star(s) = \cot^n s\Pi/2$.

We find that this leads formally to the following class of kernels,

$$k(x) = \frac{2}{\Pi}\cdot\frac{1}{(n-1)!}\,\frac{p_{n-1}\left(\frac{2}{\Pi}\log x\right)}{1-x^2} + c_n\,\delta(x-1), \tag{2}$$

where $p_n(x)$ is a polynomial of degree n in x satisfying the recurrence equation

$$p_n(x) = -xp_{n-1}(x) - n(n-1)p_{n-2}(x), \tag{3}$$

where $p_0(x) = 1$, $p_1(x) = -x$ and c_n is a constant equal to 0 if n is odd, and equal to $(-1)^{\frac{1}{2}n}$ if n is even. $\delta(x)$ is Dirac's function. It happens that $p_n(x)$ satisfies the following finite difference equation

$$f_{n+1}(x) = xf_n(x) - n(n-1)f_{n-1}(x). \tag{4}$$

The polynomials $f_n(x)$ have been investigated by Richard Kelisley and L. Carlitz in 1959 [2], [3].

In §2 we investigate the recurrence relation (3). In §3 it is proved that $p_n(x)$ satisfies the difference equation (4). In §4, the transformation for $n = 2$ is

Received June 2, 1963. The author wishes to thank Professor A. P. Guinan for suggesting this problem.