PARAMETRIZATION OF AUTOMORPHIC FORMS FOR THE HECKE GROUPS $G(\sqrt{2})$ AND $G(\sqrt{3})$

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1. Introduction. In this paper we are concerned with the parametrization of all automorphic forms on the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$. We also give formulas for the multiplier systems for these groups and every real dimension $-r$. In obtaining the form of the multiplier systems we use the results of Knopp [2] on the characters of these groups. The Hecke groups are in some sense similar to the modular group; Rademacher and Zuckerman [6], [8] carried out the parametrization of forms and multiplier system for the modular group.

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2. Preliminaries. Hecke [1] introduced the discontinuous groups $G(\lambda_n)$ generated by the transformations $w = -1/z$ and $w = z + \lambda_n$ where $\lambda_n = 2 \cos \pi/n$ and $n \geq 4$. The case $n = 3$ gives rise to the classical *modular group* $\Gamma(1)$ generated by $w = -1/z$ and $w = z + 1$. We shall be interested in the cases $n = 4$ and $n = 6$. The generators are then of the form:

$$(2.1) \qquad \begin{aligned} G(\sqrt{2}) &: Uz = z + \sqrt{2}, \qquad Tz = -1/z; \\ G(\sqrt{3}) &: Uz = z + \sqrt{3}, \qquad Tz = -1/z. \end{aligned}$$

A general element of $G(\sqrt{m})$ will be denoted by $Vz = (\alpha z + \beta)/(\gamma z + \delta)$ $(m = 2, 3)$. Let $\bar{G}(\sqrt{m})$ denote the homogeneous groups generated by the two-by-two matrices $U = (1 \sqrt{m} \mid 0\ 1)$ and $T = (0\ -1 \mid 1\ 0)$ (written in one line with a bar separating rows). Then to each element $w = Vz$ of $G(\sqrt{m})$ there corresponds two elements of $\bar{G}(\sqrt{m})$, namely $\pm(\alpha\beta \mid \gamma\delta)$. In a similar fashion we let $\bar{\Gamma}(1)$ denote the homogeneous modular group generated by $U = (1\ 1 \mid 0\ 1)$ and $T = (0\ -1 \mid 1\ 0)$.

The elements of $\bar{G}(\sqrt{m})$ $(m = 2, 3)$ can be separated into two complexes, the so-called even and odd substitutions. The even substitutions are of the form

$$(2.2a) \qquad V = (a\ b\sqrt{m} \mid c\sqrt{m}\ d), \quad ad - mbc = 1,\ a, b, c, d \text{ integers};$$

and the odd substitutions have the form

$$(2.2b) \qquad V = (a\sqrt{m}\ b \mid c\ d\sqrt{m}),\ mad - bc = 1,\ a, b, c, d \text{ integers } (m = 2, 3).$$

Let r be a real number. A function $f(z)$ meromorphic in the upper half-plane

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