GENERALIZATIONS OF AN INEQUALITY OF HARDY

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Hardy proved the following theorem [3; 240]: *If* $p > 1$, $f(x) \geq 0$ *for* $0 < x < \infty$, *and*

$$G(x) = \frac{1}{x}\int_0^x f(t)\,dt, \tag{1.1}$$

then

$$\int_0^\infty G^p\,dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p\,dx \tag{1.2}$$

unless $f \equiv 0$. *The constant is best possible.* For generalizations see references [1], [2].

Here functions are assumed to be measurable and left sides of inequalities exist when right sides do.

Generalizations will be given here of which an example in simplified form is the following.

Theorem 1. *On an open interval, finite or infinite, let* $\phi(u) \geq 0$ *be defined and have a second derivative* $\phi'' \geq 0$. *For some* $p > 1$ *let*

$$\phi\phi'' \geq \left(1 - \frac{1}{p}\right)(\phi')^2. \tag{1.3}$$

At the ends of the interval let ϕ *take its limiting values, finite or infinite. Then if for* $0 < x < \infty$, *the range of values of* $f(x)$ *lie in the closed interval of definition of* ϕ *and if* G *is as in Hardy's theorem*

$$\int_0^\infty \phi(G(x))\,dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty \phi(f(x))\,dx \tag{1.4}$$

unless $\phi(f) \equiv 0$.

The example $\phi(u) = u^p$, $u \geq 0$, shows the constant is best possible.

Proof. Let

$$\psi(u) = (\phi(u))^{1/p} \geq 0.$$

Then by (1.3) $\psi''(u) \geq 0$ where $\psi(u) > 0$. Hence $\psi(u)$ is convex. Thus by Jensen's inequality

$$\psi\left(\frac{1}{x}\int_0^x f(t)\,dt\right) \leq \frac{1}{x}\int_0^x \psi(f(t))\,dt. \tag{1.5}$$

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