FORMAL SOLUTION OF NONLINEAR SIMULTANEOUS EQUATIONS: REVERSION OF SERIES IN SEVERAL VARIABLES

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1. **Introduction.** For each i, j, $1 \leq i, j \leq n$ let $w_j = z_j f_j(z_1, \cdots, z_n)$ and $z_i = w_i h_i(w_1, \cdots, w_n)$ where the f_j and h_i are analytic and nonzero at the origin. The coefficients of the h_i are given explicitly in Theorem 3 in terms of the coefficients of the f_j. They are obtained as follows. A combinatorial argument leads to Lemma 2, which makes possible, in Lemmas 4 and 5, a simplification of certain sums of derivatives arising out of I. J. Good's [3] recent generalization of Lagrange's formula. This is enough to prove Theorem 1, a generalization of the formula for reversion of series [4; 406–407]. The explicit representations of the coefficients then follow from a result [2, Theorem 1] in the algebra of formal power series. The notations, conventions and definitions of [2, §2] are used everywhere below, usually without specific mention. This means, among other things, that lower case Greek letters other than $\zeta = (z_1, \cdots, z_n)$, $\tau = (t_1, \cdots, t_n)$ or $\omega = (w_1, \cdots, w_n)$ always represent elements of the set F_n of n-partite numbers. In particular, $\beta = (b_1, \cdots, b_n)$, $\mu = (m_1, \cdots, m_n)$ and $\xi = (x_1, \cdots, x_n)$ are elements of F_n.

2. **Combinatorial preliminaries.** Let $N(\beta) = \{j : b_j = 0\}$, $U(\beta) = \{j : b_j = 1\}$, $\mu^{-\beta} = 1/\mu^{\beta}$ (see [2, §2]). If $\beta \leq \nu = (1, 1, \cdots, 1)$, let $X(\beta) = \{\tau : \beta \leq \tau \leq \nu\}$ and let $Q(\beta, \mu) = \{\Gamma = \{(j, \gamma(j)) : \gamma(j) = (g_{j1}, \cdots, g_{jn}), j \in U(\beta)\} : \sum_{j \in U(\beta)} \gamma(j) = \mu\}$. $Q(\beta, \mu)$ is, to all intents and purposes, the set of ordered partitions [1] of μ into $|\beta|$ parts, some of which may be zero. If $\mu \leq \beta \leq \nu$, let $W(\beta, \mu) = \{\Delta = \{(j, \delta(j)) : \delta(j) = (d_{j1}, \cdots, d_{jn}), j \in U(\beta)\} \in Q(\beta, \mu)$: If $\theta < \chi \leq \beta$, then $\sum_{j \in U(\chi)} \delta(j) \neq \chi\}$. Evidently

Lemma 1: *If $\varphi \leq \nu$, the coefficient of ζ^{λ} in*

$$\prod_{j \in U(\varphi)} \Big(\sum_{\mu} {}_j c_{\mu} \zeta^{\mu}\Big) \quad \textit{is} \quad \sum_{\Delta \in Q(\varphi, \lambda)} \prod_{j \in U(\varphi)} {}_j c_{\delta(j)}\,.$$

For the remainder of this section λ and Γ are fixed, $\lambda \leq \nu$, $\Gamma = \{(j, \gamma(j)) : j \in U(\nu)\} \in Q(\nu, \nu - \lambda)$. Let $S(\Gamma) = \{\beta \in X(\lambda)$: if $j \in N(\beta)$, there is some $k = k_{\beta}(j) \in N(\beta)$ such that $\epsilon(k) \leq \gamma(j)\}$. Let $\sigma = (s_1, \cdots, s_n)$ where $s_k = \min\{b_k : \beta \in S(\Gamma)\}$ for each k, $1 \leq k \leq n$. If $k \in N(\sigma)$ and $D = \{\beta \in S(\Gamma) : k \in N(\beta)\}$, let $A(k) = \bigcap_{\beta \in D} N(\beta)$ and $A^* = \{A(k) : k \in N(\sigma)\}$. Note that one element $T \in A^*$ can correspond to many indices k. With this understanding no harm can come from convenient notations such as $A^* = \{A(k) : k \in N(\sigma)\}$ or $N(\beta) = \bigcup_{A(k) \in B^*} A(k)$.

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