DISCRETE POTENTIAL AND BOUNDARY VALUE PROBLEMS

By Charles Saltzer

1. **Introduction.** The study of boundary value problems for the Laplace difference equation has been restricted primarily to those dealing with finite regions [5]. The results obtained concerning the fundamental solution of the Laplace difference equation or discrete potential by Duffin [2], Duffin and Shelly [3], McCrea and Whipple [8], Stöhr [10], Saltzer [9] and others make it possible to consider boundary value problems for infinite regions. Uniqueness and existence theorems for the discrete boundary value problems in three or more dimensions for finite and infinite regions are given and the structure of the class of solutions is characterized in terms of discrete potentials. In addition, the discrete analogues of the integral equations of potential theory are obtained which reduce the discrete boundary value problems for both finite and infinite regions to systems of linear equations whose order is equal to the number of of boundary points. For exterior problems this reduction has an advantage for numerical calculations since the infinite region need not be replaced by a finite region. In addition, the present formulation is being used to obtain estimates for the discretization error of some boundary value problems and may lead to methods for improving the degree of approximation.

In the first section the combinatorial form of Stoke's Theorem (Eckmann [4]) is extended to infinite 1-complexes. Then the operator which transforms a discrete "mass" density into a discrete potential is studied. In the following section a generalized boundary value problem is formulated which includes all the boundary value problems as special cases, and eigenvalue bounds are obtained for the associated operators. Uniqueness theorems for the boundary value problems are established and the discrete analogues of the integral equations of potential theory for simple and double layer potentials are derived and used to establish the existence of solutions of the boundary value problems as well as the representation of the solutions as discrete analogues of single and double layer potentials.

The terminology used in this paper is the terminology of linear operator theory. The term eigenvalue as used below designates the reciprocal of the eigenvalue as defined in integral equations. The expression adjoint homogeneous system of equations also follows the usage in linear operator theory [6], and for a system of linear simultaneous equations it denotes the transposed homogeneous system [1].

The restriction to spaces of dimension greater then two is due to the singularity

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