THE SQUARE-INTEGRABILITY OF MATRIX-VALUED FUNCTIONS WITH RESPECT TO A NON-NEGATIVE HERMITIAN MEASURE

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1. **Introduction.** Let $\mathfrak{B}$ be a σ-algebra of subsets of a space Ω, and let $\mathbf{M} = [M_{ij}]$ be a $q \times q$ non-negative hermitian-matrix-valued function on $\mathfrak{B}$ which is countably additive, i.e., each entry function M_{ij} is countably additive on $\mathfrak{B}$. We shall refer to $\mathbf{M}$ as a *non-negative hermitian-valued measure* on $(\Omega, \mathfrak{B})$. Our purpose is to settle the problem of defining the integral $\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Psi}^*$, where $\mathbf{\Phi}$, $\mathbf{\Psi}$ are $p \times q$ matrix-valued functions on Ω, so that the space $\mathbf{L}_{2,\mathbf{M}}$ of functions $\mathbf{\Phi}$ for which $\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Phi}^*$ exists becomes a Hilbert space under the inner product $((\mathbf{\Phi}, \mathbf{\Psi}))_\mathbf{M} = \tau \int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Psi}^*$, i.e., is complete. (The symbols * and τ stand for *adjoint* and *trace*, respectively.)

The case in which $\Omega = (0, 2\pi]$ or $\Omega = (-\infty, \infty)$ and $\mathbf{M}$ is the measure $\mathbf{M_F}$ induced on the Borel subsets of Ω by a $q \times q$ increasing non-negative hermitian-valued function $\mathbf{F}$ on Ω is fundamental in the theory of q-variate weakly-stationary stochastic processes, (S.P.), and has been treated by workers in this field. In 1958 Wiener and Masani [6, Part II; 112] solved the problem under the assumption that $\mathbf{F}$ is absolutely continuous on $(0, 2\pi]$ and of full rank, i.e., $\mathbf{F}' > \mathbf{0}$ a.e. (Lebesgue measure). (For $q \times q$ matrices A and B, $A > B$ and $A \geq b$ mean $A - B$ is positive Hermitian and $A - B$ is non-negative Hermitian, respectively.) In 1959 Masani [3] extended this result to the case in which $\mathbf{F}$ is absolutely continuous though not of full rank but is the spectral distribution of a non-deterministic process. Independently Rosanov [4, Theorem 5] solved the problem under the mere assumption that $\mathbf{F}$ is absolutely continuous. The problem has not yet been settled for a non-absolutely continuous function $\mathbf{F}$.

In this paper we shall solve the problem for an arbitrary measure space $(\Omega, \mathfrak{B}, \mathbf{M})$, and so dispense with the assumption of absolute continuity. The point of departure of our approach is the introduction of a matricial Radon-Nikokym derivative of $\mathbf{M}$ with respect to the non-negative real-valued *trace measure* $\tau\mathbf{M}$. In §2 we shall define and study the properties of this derivative. In §3 we shall show how its use leads to a solution of our problem. In §4 we shall define the stochastic integral. In §5 we shall show that if $\mathbf{M}$ is the spectral measure corresponding to a q-variate weakly-stationary S.P. $(\mathbf{f}_n)_{-\infty}^{\infty}$, then $\mathbf{L}_{2,\mathbf{M}}$ is isomorphic to the space $\mathfrak{S}\{\mathbf{f}_n\}_{-\infty}^{\infty}$ spanned by the vectors $\mathbf{f}_n$, $-\infty < n < \infty$. This result has been referred to in the literature, cf. e.g. [1; 596], but not really proved since the completeness of $\mathbf{L}_{2,\mathbf{M}}$ was never fully established.

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