SOME THEOREMS IN SET THEORY AND APPLICATIONS IN THE IDEAL THEORY OF PARTIALLY ORDERED SETS

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This article consists of two parts. In the first section several theorems in set theory are proved. The important results are that every inductive family of sets is closed under union of directed subfamilies (strongly inductive); and that a family closed under intersection of arbitrary subfamilies (closed), if enlarged by adjoining unions of directed subfamilies, is both closed and strongly inductive.

The results of the first section are then applied to the following problem in the theory of partially ordered sets. Given a set X and a family $\mathfrak{A}$ of subsets of X, which separates points, one may define a natural partial order in X; in terms of the family $\mathfrak{A}$, which subsets of X are ideals relative to this partial order? The answer leads to a characterization of lattices which are isomorphic to the lattice of all ideals of a partially ordered set.

This work was suggested by the article [2] of O. Frink, to whom we are further indebted for his valuable suggestions.

I. **Definitions and notation.** A family $\mathfrak{A}$ of subsets of a set X is called *inductive* if for every chain $\mathfrak{C} \subset \mathfrak{A}$, $\bigcup \mathfrak{C} \in \mathfrak{A}$. It is called *strongly inductive* if for every directed family $\mathfrak{D} \subset \mathfrak{A}$, $\bigcup \mathfrak{D} \in \mathfrak{A}$, and *closed* if for every family $\mathfrak{J} \subset \mathfrak{A}$, $\bigcap \mathfrak{J} \in \mathfrak{A}$. Given a family $\mathfrak{A}$, $\mathfrak{A}^\wedge$ is the family $\{B \mid B = \bigcup \mathfrak{B}, \mathfrak{B} \subset \mathfrak{A}, \mathfrak{B} \text{ directed}\}$. For any set $C \subset X$, $\bar{C} = \bigcap \mathfrak{A}_C$, where $\mathfrak{A}_C = \{A \mid A \in \mathfrak{A}, C \subset A\}$. If $\mathfrak{A}$ is closed, $A \in \mathfrak{A}$ is called *finitely generated* if there is a finite set $F \subset X$, such that $\bar{F} = A$.

In Maeda [3] it is proved that

Theorem 1. *Every infinite directed set D is the union of a chain of directed sets, each of cardinality less than that of D.*

From this we deduce

Theorem 2. *Every inductive family is strongly inductive.*

Proof. Assume the family $\mathfrak{A}$ to be inductive, but not strongly inductive, and let $\mathfrak{D} \subset \mathfrak{A}$ be a directed subfamily of lowest cardinality for which $\bigcup \mathfrak{D} \notin \mathfrak{A}$. Then $\mathfrak{D}$ is infinite. Hence, by Theorem 1, $\mathfrak{D}$ is the union of a chain of directed subfamilies, each of lower cardinality than that of $\mathfrak{D}$, and each with its union in $\mathfrak{A}$. Hence $\bigcup \mathfrak{D}$ is the union of the chain of unions of the subfamilies, and $\bigcup \mathfrak{D} \in \mathfrak{A}$. This contradicts the assumption.

Received February 4, 1963; in revised form, February 4, 1964. Research supported by the National Science Foundation, Grant GP2214.