COMPLETE SEQUENCES OF POLYNOMIAL VALUES

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Introduction. Let $f(x)$ be a polynomial with real coefficients. In 1947, R. Sprague [7] established the result that if $f(x) = x^n$, n an arbitrary positive integer, then every sufficiently large integer can be expressed in the form

$$(1) \qquad \sum_{k=1}^{\infty} \epsilon_k f(k)$$

where ϵ_k is 0 or 1 and all but a finite number of the ϵ_k are 0. More recently K. F. Roth and G. Szekeres [5] have shown (using ingenious analytic techniques) that if $f(x)$ is assumed to map integers into integers, then the following conditions are necessary and sufficient in order for every sufficiently large integer to be written as (1):

(a) $f(x)$ has a positive leading coefficient.
(b) For any prime p there exists an integer m such that p does not divide $f(m)$.

It is the object of this paper to determine, in an elementary manner, *all* polynomials $f(x)$ with real coefficients for which every sufficiently large integer can be expressed as (1) (cf. Theorem 4).

Preliminary results. Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers.

Definition 1. $P(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$ where ϵ_k is 0 or 1 and all but a finite number of ϵ_k are 0.

Definition 2. S is said to be *complete* if all sufficiently large integers belong to $P(S)$.

Definition 3. S is said to be *nearly complete* if for all integers k, $P(S)$ contains k consecutive positive integers.

Definition 4. S is said to be a *Σ-sequence* if there exist integers k and h such that

$$s_{h+m} < k + \sum_{n=0}^{m-1} s_{h+n}, \qquad m = 0, 1, 2, \cdots.$$

(where a sum of the form $\sum_{n=a}^{b}$ is 0 for $b < a$).

The following lemma is one of the main tools used in this paper:

Lemma 1. *Let $S = (s_1, s_2, \cdots)$ be a Σ-sequence and let $T = (t_1, t_2, \cdots)$ be nearly complete. Then the sequence $U = (s_1, t_1, s_2, t_2, \cdots)$ is complete.*

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