APPROXIMATION BY POLYNOMIALS WHOSE ZEROS LIE ON A CURVE

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1. **Introduction.** Let R be a closed set in the complex plane and let an R-polynomial be a polynomial all of whose zeros belong to R. Let D be an open set in the plane. We consider approximation in the sense of UD-convergence, that is, convergence everywhere in D which is uniform on compact subsets of D. An RD-function is a holomorphic function in D, not identically zero, which is the limit of a UD-convergent sequence of R-polynomials. Suppose that R is a curve and consider the approximation problem: Give conditions on R and D under which the class of RD-functions consists of all holomorphic functions in D, not identically zero, whose zeros (in D) belong to R. We say in this case that R is a polynomial approximation set for D (see [5; 184]). The only known result in this area is due to G. R. MacLane [7, Theorem I, 461]. It asserts that every simple closed curve of finite length is a polynomial approximation set for its interior. It is our purpose to extend MacLane's theorem to more general curves. As a by-product we obtain a somewhat simpler proof of MacLane's result.

In §2 we introduce the principal idea used in this paper, that of an equilibrium family. The approximation problem for Jordan domains D with R the boundary of D is discussed in §§3 through 5. In §6 we give analogous results for some simple unbounded sets.

2. **Definition of equilibrium families.** We assume throughout that the origin belongs to D. An *equilibrium family* is a family E of sequences of points on R, $\{z_{n,\nu}\}_{\nu=1}^{n}$, $n = n_1, n_2, \cdots, n_i \to \infty$, such that $\prod_{\nu=1}^{n} (1 - z/z_{n,\nu}) \to 1$ as $n \to \infty$, in the sense of UD-convergence. The equilibrium families most useful to us will consist of points which are "well distributed" over R.

Let $H(D)$ be the class of functions holomorphic and zero free in D which are 1 at the origin.

3. **Dense subsets of $H(D)$ where D is a Jordan domain.** We suppose that D is a simply connected bounded open set and that its boundary, which we denote by Γ, is a Jordan curve. The special case of a disc is discussed in [4].

Let D^* denote the complement of $D + \Gamma$. Let $z = \Phi(w)$, $\Phi(0) = \infty$, $\Phi(1) = z_0$

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