SUBLINEAR FUNCTIONS OF MEASURES AND VARIATIONAL INTEGRALS

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The original purpose of this paper was to establish that certain non-parametric variational integrals may be considered as measures on the domain of definition of the admissible functions. This was accomplished by exhibiting an explicit formula for the functional involved, the formula containing within itself, in the case of the surface area integral, a proof both of Tonelli's celebrated theorem on Lebesgue area and of a less well known but deeper result of Verchenko. It soon became apparent, however, that the basic techniques were considerably more general, and could, in fact, be phrased entirely apart from variational calculus simply as a method of generating measures in terms of other measures. We shall follow this more abstract course in the paper, and only turn directly to variational calculus in the final section.

Consider a countably additive set function μ on a σ-ring R of subsets of a basic set S. The function μ will be supposed to have its values in a real Banach space $\mathfrak{X}$, which for particular applications may be the real numbers R^1, or the Euclidean number space R^n. Let $\mathfrak{F}(p)$ be a bounded sublinear functional on $\mathfrak{X}$ into the reals. By this we mean that for all p, q in $\mathfrak{X}$

$$\mathfrak{F}(p + q) \leq \mathfrak{F}(p) + \mathfrak{F}(q),$$

while at the same time

$$\mathfrak{F}(\lambda p) = \lambda \mathfrak{F}(p)$$

if λ is real and positive. Furthermore, there should exist a positive constant C such that

$$\mathfrak{F}(p) \leq C\ |p|,$$

where $|p|$ denotes the norm of p in $\mathfrak{X}$. It is evident that the boundedness of $\mathfrak{F}$ implies that $\mathfrak{F}$ is continuous, and, conversely, if $\mathfrak{F}$ is continuous, then it is bounded.

We can now define a new set function $\mathfrak{F}\mu$ on R by the relation

$$\mathfrak{F}\mu(E) = \sup \sum_1^N \mathfrak{F}(\mu(E_i)) \qquad E \in \mathsf{R}$$

where the supremum is taken over all finite partitions $E = \cup E_i$ of E with $E_i \in \mathsf{R}$. This set function is in some ways analogous to the total variation of μ;

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