OPERATIONAL REPRESENTATIONS FOR THE LAGUERRE AND OTHER POLYNOMIALS

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1. **Introduction.** Recently Burchnall [5], Carlitz [6], Chak [8], and Gould and Hopper [9] considered some differential operators which they used either to derive already known formulas or to obtain new ones. In particular, Chak gives the operational formula

$$L_n^{(\alpha)}(x) = x^{-\alpha-n-1}e^x(x^2D)^n(x^{\alpha+1}e^{-x}), \qquad D = d/dx,$$

as a special case of a certain generalization of the Laguerre polynomials. Gould and Hopper employed their operator to obtain a generalization of the Hermite polynomials.

In this paper we define the operator

$$\theta = x(1 + xD) \tag{1.1}$$

and thus

$$\theta^n\{x^{\alpha+k}\} = (\alpha + k + 1)_n x^{\alpha+k+n} \tag{1.2}$$

where k is an integer, n a non-negative integer, and α is arbitrary. This operator is essentially that of Chak and is closely related to those employed by Carlitz, and Gould and Hopper. We find it useful in deriving and generalizing some known formulas involving some of the classical orthogonal polynomials. As a striking example we have given three very simple and brief proofs of the Hardy-Hille formula (5.5). We also show how apparently different generating functions can be obtained from one another. For example, we derive the generating function (4.2) for the Laguerre polynomials by operating on the function $x^\alpha e^{-x}$ or on the elementary expansion

$$x^\alpha(1 - x)^{-\alpha-1} = \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k}{k!} x^{\alpha+k}.$$

As another example we show in §7 how the generating function for the Sister Celine polynomials follow from that of the Legendre polynomials.

Some bilinear generating functions are obtained in a simple and direct manner. As an example, we mention the two formulas (5.4) and (6.17) due to Weisner that have received some attention recently. Formulas (3.18), (4.8), (4.9), (5.6), (6.13), (6.18), (6.19), (6.20), and (7.6) are believed to be new. The operational representations for the Laguerre, Jacobi, Legendre and other polynomials are also new.

In the sequel we shall use the Greek letters θ, ϕ to denote the operator defined

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