THE BOUNDARY OF A ONE-PARAMETER GROUP IN A SEMIGROUP

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1. **Introduction.** In studying topological semigroups on the plane, it was found that several critical questions concerned the nature of the boundary of a one-parameter group ([2], [3], [4]). Preliminary work on semigroups in higher dimensions suggests that these questions will remain critical. Two such questions are the following: (1) If the boundary of a one-parameter group P is non-empty, must it contain an idempotent? Indeed, must it be compact? (2) If P contains an idempotent e in its boundary, when must $P^- = P \cup \{e\}$? In [2], it was shown that if P^- is embeddable in the plane as a closed subset, then the answer to both parts of (1) is affirmative and that $P^- = P \cup \{e\}$ if and only if e is a zero for P. Assuming always that P^- is locally compact, we show that, in general, if the boundary of P contains a zero 0, then $P^- = P \cup \{0\}$ (Theorem 1). It follows that if P contains a compact ideal I in its boundary, then $P^- = P \cup I$. It is then not difficult to show that if the boundary of P is not empty, then it is compact if and only if it is a group (Corollary 2). We show further that if P is a one-parameter group of non-singular complex matrices, then its boundary is compact. However, we obtain, in the final section, an example of a one-dimensional, locally compact metric semigroup S which contains an identity and no other idempotents as well as a dense, one-parameter subgroup. Algebraically, S is the cartesian product of the additive group of real numbers and the additive semigroup of non-negative integers.

What "reasonable" restrictions force the boundary of a one-parameter group to be compact are not now entirely clear. In view of Theorem 2, it is natural to ask whether the example of §3, or one like it, can be a semigroup of operators on a Hilbert space. We have given only passing thought to this question.

2. **Conditions for compactness of the boundary.** In general, the closure of a set A is denoted by A^- and its boundary by $F(A)$. A zero for a semigroup T is an element 0 (possibly in a larger semigroup) such that $0t = t0 = 0$ for all $t \in T$. To say that two topological semigroups are isomorphic means that there is a function between them which is both an algebraic isomorphism and a homeomorphism. To say that they are algebraically isomorphic means only that there is a function satisfying the first condition.

We show first that if a one-parameter group P has a zero 0 in its boundary and P^- is locally compact, then $P^- = P \cup \{0\}$. The proof follows very closely the proof of an analagous result in [2] where it was assumed that P^- is embedded in the plane. Only two changes in that argument must be made, one of these

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