A UNIQUENESS THEOREM FOR THE HELMHOLTZ EQUATION

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1. **Introduction.** The present paper establishes the following uniqueness theorem for the integral value problem, [2], for the Helmholtz equation.

Theorem. *Let $u(x_1, x_2)$ be an everywhere twice continuously differentiable solution of*

$$\partial^2u/\partial x_1^2 + \partial^2u/\partial x_2^2 + u = 0 \tag{1.1}$$

and satisfy the integral condition

$$\lim_{R\to\infty} \int_0^{2\pi} \left| \int_0^R u(x_1, x_2)\, dr \right| d\theta = 0, \quad \text{where} \quad x_1 + ix_2 = re^{i\theta}. \tag{1.2}$$

Then $u \equiv 0$.

Using the inequality of Schwarz the theorem yields the following corollary which was first proved by P. Hartman and C. Wilcox in their comprehensive paper, [1], on the Helmholtz equation.

Corollary. *The theorem remains true when the condition* (1.2) *is replaced by*

$$\lim_{R\to\infty} \int_0^{2\pi} \left| \int_0^R u(x_1, x_2)\, dr \right|^2 d\theta = 0. \tag{1.3}$$

Our method of proof resembles that used by F. Rellich, [3], to show the uniqueness of the radiation problem for the 3-dimensional Helmholtz equation. Therefore, the mean value equation for (1.1) assumes an essential role in this investigation and it enables us to show, under the assumption (1.2), that u, together with all its partial derivatives, vanishes at the origin. Hence, as any twice continuously differentiable solution of (1.1) is necessarily analytic in the variables (x_1, x_2), it follows that u vanishes everywhere.

The author expresses his gratitude to the referee for showing that our proof, originally used to treat the corollary, would yield the theorem.

2. **The mean value equation.** If $u(x_1, x_2)$ is any twice continuously differentiable solution of (1.1), then

$$v(r) \equiv \frac{1}{2\pi} \int_0^{2\pi} u(x_1, x_2)\, d\theta \tag{2.1}$$

is a solution of the Bessel differential equation

$$r d^2v/dr^2 + dv/dr + rv = 0. \tag{2.2}$$

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