EXTENSION OF RESULTS CONCERNING RINGS IN WHICH SEMI-PRIMARY IDEALS ARE PRIMARY

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1. **Introduction.** An ideal A of a ring R is said to be semi-primary if $\sqrt{A}$ is a prime ideal. A ring R is said to satisfy Condition (*) if every semi-primary ideal of R is primary. In a previous paper [1], the author investigated domains with identity and noetherian rings with identity satisfying (*).

The present paper considers rings without identity satisfying (*). Theorem 7 of §4 represents the most significant result of the paper. In this theorem, a complete classification of rings satisfying (*) is obtained.

All rings considered in this paper will be assumed to be commutative and to contain more than one element. The terminology used is that of van der Waerden [5].

2. **Properties of rings satisfying** (*). We shall consider in this section some properties of a ring R satisfying (*). The proofs of some theorems in [1], given for the case when R contains an identity, carry over with little or no change to the case when R has no identity. Such theorems will be indicated by (P) and proofs will not be repeated. Only those results of [1] which are actually used in obtaining Theorem 7 of §4 are restated.

DEFINITION. A ring S is said to have *dimension* n or to be *n-dimensional* if there exists a strictly ascending chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of proper prime ideals of S, but no such chain of $n + 2$ proper prime ideals exists in S.

We list the following properties of a ring R satisfying (*).

PROPERTY 1. *Any homomorphic image of R satisfies* (*).

PROPERTY 2. *If A and B are ideals of R with $A \subseteq B \subseteq \sqrt{A}$ and if A is primary for $\sqrt{A}$, then B is primary for $\sqrt{A}$.*

THEOREM 1. *If P is a nonmaximal proper prime ideal of a ring R satisfying* (*) *and if Q is primary for P, then $Q = P$.*

Proof. Since P is nonmaximal and proper, there exists an ideal A of R such that $P \subset A \subset R$. If $a \in A - P$ and if $p \in P$, then $Q \subseteq Q + (ap) \subseteq P$. By Property 2, $Q + (ap)$ is primary for P. If $s \in R - A$, then $sap \in Q + (ap)$. Since $a \notin P$, $sp \in Q + (ap)$. Then for some $q \in Q$, $r \in R$, and $d \in Z$, $sp = q + rap + dap$. Therefore $p(s - ra - da) \in Q$. Because $s \notin A$, $s - ra - da \notin P \subset A$. Hence $p \in Q$ and $P = Q$ as the theorem asserts.

COROLLARY 1.1 (P). *If ring R satisfies* (*), *if P_1 and P_2 are proper prime ideals of R with $P_1 \subset P_2$, and if Q is primary for P_2, then $P_1 \subset Q$.*

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