SYMMETRIC, CONTINUOUS, AND SMOOTH FUNCTIONS

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Introduction. A real-valued function f defined in a neighborhood of $I_0 = [0, 1]$ will be termed *symmetric* on I_0 if for each $x \varepsilon I_0$, $f(x+h) + f(x-h) - 2f(x) = o(1)$ as $h \rightarrow 0$, and following Zygmund [9] f is said to be *smooth* on I_0 if for each $x \varepsilon I_0$, $f(x + h) + f(x - h) - 2f(x) = o(h)$ as $h \rightarrow 0$. Symmetric and smooth functions have found many applications in trigonometrical series [10, I–II], and thus have attracted a great deal of attention. The first part of the paper is devoted to the study of the relationship between continuity and symmetry, and, in particular, to present properties that symmetric and continuous functions have in common. S. Mazurkiewicz [4] has shown that a bounded symmetric function is continuous a.e., and H. Auerbach has simplified the proof in [1] where he has also shown that a symmetric and summable function is of the 1-st Baire class. Among other results it is shown that in Auerbach's theorem "summability" can be weakened to "measurability".

It is known [9] that a continuous and smooth function on I_0 is differentiable on a set which is of the power of the continuum in each interval. In the second half of the paper it is seen that "continuity" can be replaced by "measurability" by showing that a measurable smooth function is continuous on a dense open set.

Let E be the set of points at which a measurable and smooth function f is differentiable. If f is continuous, then f' possesses the Darboux property on E [9], even though, as it is known, E can be of measure zero. If one only assumes measurability of f instead of continuity, it will be seen that f' need not have the Darboux property on E. However, if E is "small" in the sense that $|E \cap I| < |I|$ for every interval $I \subset I_0$, then f' does have the Darboux property on E. This curious feature is reflected in the fact that the "smallness" of E forces f to be highly oscillatory in the sense that the set A of points at which the upper derivate is $+\infty$ and the lower one is $-\infty$ is residual in I_0. Further, if A is of the 1-st category, then E is "large," i.e., there is a dense open set G in I_0 such that f' exists a.e. on G. The paper concludes with an extension (to measurable symmetric functions) of a known theorem on derivates of continuous functions.

Notation. Let R be the set of real numbers. For $A \subset R$, A^0 is the interior of A and $|A|$ denotes the Lebesgue measure of A. For $f : I_0 \rightarrow R$, we will abbreviate by $C(f)$ the set of points in I_0 at which f is continuous, and by $E(f)$ the set of points in I_0 at which f has a finite derivative. Finally, we recall that a set $A \subset X$ is called *residual* in X if $X - A$ is of the 1-st category [2].

Symmetric functions. Let S be the class of symmetric functions on I_0, and let S_0 be the subclass of all measurable $f \varepsilon S$. The following lemma will prove important in the sequel.

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