# SOME SOLUTION-PRESERVING OPERATORS FOR A PARTIAL DIFFERENTIAL EQUATION

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1. **Canonical forms.** Consider the differential equation

$$L[w] \equiv Aw_{xx} + Bw_{xy} + Cw_{yy} + Dw_x + Ew_y = 0 \tag{1.1}$$

where $A, \cdots, E$ are real constants with $A, B, C$ not all zero. If $w$ is a solution, and has a sufficient number of derivatives, then finite linear combinations with constant coefficients:

$$H[w] \equiv \sum_{i,k} a_{ik} D_x^i D_y^k w \tag{1.2}$$

are also solutions. Combinations (1.2) are thus solution-preserving (hererafter abbreviated s.p.) for equation (1.1). It seems to have gone unnoticed that there are also s.p. operators of form (1.2) where the coefficients $a_{ik}$ need not be constants. It is such operators that are investigated here. This is done by reducing (1.1) to certain "canonical" forms by means of suitable real, non-singular linear transformations, and examining each such form separately

Lemma 1.1. *There exist real transformations with constant coefficients:*

$$(x, y) \to (ax + by, cx + dy) \qquad (ad - bc \neq 0) \tag{1.3}$$

*that carry equation* (1.1) *into one of the following forms:*

$$w_{xy} - w_x - w_y = 0, \tag{1.4}$$

$$w_{xy} = 0, \tag{1.5}$$

$$w_{xy} - w_y = 0 \quad \textit{if} \quad B^2 - 4AC > 0; \tag{1.6}$$

$$\Delta w + w_y = 0 \qquad (\Delta = \textit{Laplace operator}), \tag{1.7}$$

$$\Delta w = 0 \quad \textit{if} \quad B^2 - 4AC < 0; \tag{1.8}$$

$$w_{xx} - w_y = 0, \tag{1.9}$$

$$w_{xx} = 0, \tag{1.10}$$

$$w_{xx} - w_x = 0 \quad \textit{if} \quad B^2 - 4AC = 0. \tag{1.11}$$

*Moreover no one of equations* (1.4) *to* (1.11) *is reducible to another of these by a real non-singular transformation* (1.3).

The proof is carried out by straightforward computations that we omit.

In each of the important cases (1.4), (1.7), (1.9) we shall show that all s.p.

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