ENDOMORPHISMS OF MINIMAL SETS

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1. **Introduction.** Let (X, T) be a transformation group. A continuous map φ of X to X is called an *endomorphism* if $\varphi(x)t = \varphi(xt)$, for all $x \in X$, $t \in T$. If φ is one-one, φ is said to be an *automorphism*. We denote the endomorphisms of (X, T) by $H(X)$ or H, and the automorphisms of (X, T) by $A(X)$ or A. If $A(X) = H(X)$, the transformation group (X, T) is said to be *coalescent*.

A non-empty subset M of X is said to be a *minimal set*, if, for every $x \in M$, the orbit closure $(xT)^- = M$. In this paper, we will be concerned with the case where (X, T) is itself a minimal set. The phase space X will be assumed to be compact Hausdorff. With regard to the questions considered here, the topology of T does not play an important role, and we may assume that T is simply a group of self homeomorphisms of X.

We observe that every endomorphism of a minimal set (X, T) is onto, so that $A(X)$ is a group and $H(X)$ is a semigroup.

Endomorphisms of minimal sets are closely connected with the enveloping semigroup of a transformation group and its minimal right ideals, [4]. The basic properties of the enveloping semigroup are summarized in §2. In §3, a quasi-ordering in a minimal set is defined in terms of a minimal right ideal of the enveloping semigroup. This quasi ordering gives a criterion for the existence of endomorphisms of the minimal set (Theorem 1). The minimal right ideals are themselves minimal sets, and their endomorphisms are determined in §4 (Theorem 3). Theorem 4 provides an intrinsic characterization of the minimal right ideals—that is, a characterization solely in terms of transformation group properties. In §5, we show that any endomorphism of a minimal set is induced by an endomorphism of a minimal right ideal in its enveloping semigroup (Theorem 6). The endomorphisms of a special class of minimal sets, the proximally equicontinuous ones, are studied in §6. The concluding section shows how noncoalescent minimal sets may be obtained.

2. **The enveloping semigroup of a transformation group.** In this section, (X, T) may be any transformation group with compact Hausdorff phase space X. As is customary, let X^X denote the set of all functions from X to X, provided with the topology of pointwise convergence, and consider T as a subset of X^X. Let $E(X)$ or E denote the closure of T in X^X. E is a compact Hausdorff space. If $\xi_1, \xi_2 \in E$, and if $\xi_1\xi_2$ is defined by $x(\xi_1\xi_2) = (x\xi_1)\xi_2 (x \in X)$, then $\xi_1\xi_2 \in E$. That is $EE \subset E$; in particular, $ET \subset E$. Therefore, we may consider (E, T) as a transformation group, whose phase space E admits a semigroup structure.

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