RICCATI ALGEBRAS

By J. J. Levin* and S. S. Shatz

Introduction. In this paper we study an explicit class of algebras and the Jordan structure on them. They arose by applying a general construction due to Markus [7] to the real homogeneous matrix Riccati equation.

$$\frac{dX}{dt} = X\Gamma X, \tag{0.1}$$

where Γ is a given $m \times n$ matrix over R (the reals) and X is an $n \times m$ matrix. The Riccati equation is relevant to a number of problems in differential equations; a short bibliography may be found in [6].

In §1 we show that the (Markus) algebra associated with (0.1) is the algebra $\mathcal{R}$ of all $n \times m$ matrices over R with multiplication defined by

$$A \cdot B = \tfrac{1}{2}(A\Gamma B + B\Gamma A) \qquad (A, B \in \mathcal{R}). \tag{0.2}$$

With the above as motivation, we replace R by an arbitrary commutative, associative ring with unity, K, in which 2 is a unit and make the

Definition. *The associative algebra $\mathcal{A}$ over K, associated with the $m \times n$ matrix Γ, is the K-module of all $n \times m$ matrices over K with multiplication defined by*

$$A * B = A\Gamma B \qquad (A, B \in \mathcal{A}). \tag{0.3}$$

We denote by $\mathcal{R}$ the Jordan algebra $\mathcal{A}^+$ [3] associated with $\mathcal{A}$, so that $\mathcal{R}$ is the same K-module as $\mathcal{A}$ but with multiplication given by (0.2). Because of its origin, we refer to $\mathcal{R}$ as the Riccati algebra over K associated with Γ. If $n = m$ and $\Gamma = E_n$ (the unit $n \times n$ matrix), then $\mathcal{A}$ reduces to the classical algebra of all $n \times n$ matrices over K and $\mathcal{R}$ becomes the classical Jordan algebra of all $n \times n$ matrices over K [1].

Jacobson [5] has considered the multiplications (0.2), (0.3) (for $n = m = 3$ with a particular Γ) for entirely different purposes.

If K is a field, much can be said about the structure of $\mathcal{A}$. However, this is a considerably stronger hypothesis than is required to obtain detailed information. In §2 we introduce three weaker hypotheses (α), (β), (γ) which turn out to be very natural for the study of $\mathcal{A}$ and $\mathcal{R}$. Then we discuss relations among them and investigate their relevance for the idempotent structure of $\mathcal{A}$. For example, two of them become equivalent when K is a local noetherian ring. Also, if there exists a $Q \in \mathcal{A}$ such that $\Gamma Q \Gamma = \Gamma$ (our hypothesis (β)),

Received September 15, 1962.

* MIT, Lincoln Laboratory. Operated with support from U. S. Army, Navy, and Air Force.