STRUCTURE THEOREMS FOR CERTAIN OPERATOR ALGEBRAS

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By an *operator* on a complex vector space V we mean a linear transformation mapping V into V. We say that a subspace U of V *reduces* an operator A on V if $AU \subset U$. By an *algebra* of operators on V we mean a nonzero collection of operators closed under products and linear combinations. The *commutant* of an operator algebra M is the algebra of all operators on V which commute with every operator in M. Let M be an algebra of operators on a vector space V, let N be an algebra of operators on a vector space U and let n be a (possibly transfinite) cardinal number. We say that M is an *n-fold copy* of N if V is the direct sum of n subspaces V_α, each reducing every operator in M, for which there is a 1-1 linear transformation T_α mapping U onto V_α such that for any operator A in M there is an operator B in N for which $T_\alpha B T_\alpha^{-1}$ is the contraction of A to V_α for all α, and conversely, for any operator B in N there is an operator A in M for which $T_\alpha B T_\alpha^{-1}$ is the contraction of A to V_α for all α. An algebra of operators is said to be *finite dimensional* if it is a finite dimensional complex vector space.

We say an operator P on V is a *projection* if $P^2 = P$ [3; 73]. Two projections P and Q on V are said to be orthogonal ($P \perp Q$) if $PQ = QP = 0$; we say P is *less than* $Q (P < Q)$ if $PQ = QP = P$. The reader can immediately verify that if $P < Q$ and $Q < R$, then $P < R$ for any projections P, Q and R. Also if $P < Q$ and $Q \perp R$, then $P \perp R$. If $P \perp Q$, then $P + Q$ is a projection and $P < P + Q$, $Q < P + Q$.

We say that an operator A is *algebraic* if there exists a nonzero polynomial p with complex coefficients for which $p(A) = 0$ [4; 38]. The *degree* of an algebraic operator A is the degree of the minimum polynomial of A. An operator A is *nilpotent* if there is a positive integer n for which $A^n = 0$.

In the present paper we will present several analogues for algebras of operators of the following proposition concerning single operators [4, Lemma 14].

PROPOSITION. *Let A be an operator on a complex vector space V such that for each vector z in V the linear manifold V_z spanned by all the vectors $z, Az, A^2z, \cdots$ is finite dimensional and $\sup_{z \varepsilon V} \dim V_z < \infty$. Then A is algebraic and V is the direct sum of finite dimensional subspaces each reducing A.*

One might conjecture that if M is an operator algebra such that Mz is finite dimensional for each z in V and if $\sup_{z \varepsilon V} (\dim Mz) < \infty$, then M must be finite dimensional. To see that this is false consider a finite dimensional subspace U of an infinite dimensional vector space V and let M consist of all

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