ZETA FUNCTIONS OF NON-MAXIMAL ORDERS IN RATIONAL SEMISIMPLE ALGEBRAS

BY W. E. JENNER

The zeta series for a rational semisimple algebra can be constructed for non-maximal as well as maximal orders. The purpose of this note is to establish the domain of convergence in the non-maximal case. This is of interest due to the occurrence of such orders in the theory of complex multiplication.

Suppose A is a semisimple algebra of dimension n over the field Q of rational numbers. An *order* of A is a subring $\mathfrak{O}$ of A which is a finitely-generated module of rank n over the ring Z of rational integers, and which contains the unit element of A. Left and right ideals of $\mathfrak{O}$ are defined as usual (cf. [1], [3]). It is known that under the present assumptions every order $\mathfrak{O}$ can be embedded in a maximal order $\mathfrak{O}^*$. The *conductor* $\mathfrak{F}$ of $\mathfrak{O}$ in $\mathfrak{O}^*$ is the sum of all two-sided integral $\mathfrak{O}^*$-ideals contained in $\mathfrak{O}$. The *zeta series* of $\mathfrak{O}$ is defined by $\zeta(s) = \sum (N\mathfrak{a})^{-s}$ where $\mathfrak{a}$ ranges over the set of integral left ideals of $\mathfrak{O}$ and where $N\mathfrak{a}$ is the norm of $\mathfrak{a}$, that is the index of $\mathfrak{a}$ in $\mathfrak{O}$ as an additive group. The variable s is taken complex, and the problem we consider is that of determining for what values of s the zeta series converges. In the classical case of maximal orders, it is known that the series converges absolutely for Re $(s) > 1$ (cf. [1]). It will be shown that the same result holds in the non-maximal case. Our proof uses in an essential way the result for maximal orders and so, of itself, does not provide any new information of much depth. On the other hand, the actual question does not seem to have been considered in the literature, and its answer provides new information which is of interest. In particular it shows that, in a certain reasonable sense, there are not "too many" integral left ideals of given norm in a non-maximal order.

THEOREM. *Let $\mathfrak{O}$ be an arbitrary order in a semisimple algebra of finite dimension over the field of rational numbers. Then the zeta series $\zeta(s) = \sum N\mathfrak{a}^{-s}$ converges absolutely for $Re(s) > 1$.*

Proof. Two left ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{O}$ are said to be equivalent if there is a regular element $\lambda \in A$ such that $\mathfrak{a} = \mathfrak{b}\cdot\lambda$. It is known that the number of resulting equivalence classes is finite (cf. [4]). Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_h$ be a complete system of representatives of these classes. These left ideals can be chosen integral, and it will be assumed that such is the case. Now let c be a non-zero element of Z such that $c\mathfrak{O} \subset \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_h$. Then $c\mathfrak{F} \subset \mathfrak{a}_i \subset \mathfrak{O} \subset \mathfrak{O}^*$ for $i = 1, \cdots, h$. We now let $\mathfrak{a}$ range over the set of integral left ideals of $\mathfrak{O}$ and consider the mapping $\varphi : \mathfrak{a} \rightarrow \mathfrak{O}^*\mathfrak{a}$ into the set of integral left ideals of $\mathfrak{O}^*$. The

Received October 6, 1962.