A NOTE ON UNIFORMLY LOCALLY CONNECTED OPEN SETS

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The theorems below describe properties of ulc^n open subsets of a compact Hausdorff space, n a non-negative integer. The first answers a question posed by R. L. Wilder [2; 382], concerning the local connectedness of the closure of ulc^n open sets, by extending a result he has obtained to non-metric spaces and to coefficients in an elementary compact group as well as a field. (An elementary compact group is a finite direct sum of finite cyclic groups and copies of the reals mod 1.) The second shows that a ulc^n open set and its closure have isomorphic homology groups in dimensions $\leq n$. Both of these results are based on a theorem due to E. E. Floyd [1, 2.3].

For a compact space X, $H_n(X)$ will denote the n-dimensional Čech homology group of X, which will be the reduced group when $n = 0$. Coefficients will be in a field or compact abelian group. A closed covering, α, of a space X will mean a finite ordered collection of closed sets whose interiors cover X. $H_n(\alpha)$ will be used to denote the n-dimensional homology group of the nerve of α, and π_α assigns to each element of $H_n(X)$ its coordinate in $H_n(\alpha)$. If $A \subset B$, where A, B are compact, I_{AB} will denote the homomorphism $H_n(A) \to H_n(B)$ induced by inclusion. $H_n(A; B)$ represents the image of I_{AB}.

If α, β are finite collections of closed sets with β refining $\alpha(\beta > \alpha)$, then $\pi_{\beta\alpha}$ denotes the natural projection homomorphism $H_n(\beta) \to H_n(\alpha)$. If $\beta > \alpha$, then β *n-refines* α, ($\beta >^n \alpha$) if and only if for each $B \in \beta$ there is an $A \in \alpha$, $B \subset A$, so that $H_j(B; A) = 0$, for $j = 0, 1, \cdots, n$. β *strongly n-refines* α ($\beta \gg^n \alpha$) provided there is a projection $\pi : \beta \to \alpha$ so that for $B_{i_1}, \cdots, B_{i_q} \in \beta$, $H_j(B_{i_1} \cap \cdots \cap B_{i_q}; \pi B_{i_1} \cap \cdots \cap \pi B_{i_q}) = 0$, for $j \leq n$. X is a compact Hausdorff space throughout.

Definition. Let U be an open subset of X. U is said to be ulc^n if and only if for each $x \in \bar{U}$ and open set V containing x, there is an open set W containing x so that $W \subset V$, and for each compact set $A \subset W \cap U$ there is a compact set B in $V \cap U$, with $A \subset B$, so that $H_j(A; B) = 0$, all $j \leq n$.

In Theorem (3.5) of [1] the proof of the implication $(a) \to (b)$ remains valid without assuming the space compact as long as B is compact. Using this fact it can be shown that Wilder's definition of ulc^n for an open set U [2; 292, 299] is equivalent to the above in case the coefficients are in a field or elementary compact group.

Lemma 1. *If U is a ulc^n, open subset of X, and if α is a closed covering of $\bar{U}$, then there is a closed covering β of $\bar{U}$, $\beta > \alpha$, so that if B is a compact subset of U, then there is a compact subset A of U, $B \subset A$, so that $\beta \cap B \gg^n \alpha \cap A$.*

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